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FIREFIGHTING AS A STRATEGIC GAME

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Abstract *The Firefighter Problem was proposed in 1995 as a deterministic discrete-time model for the spread and containment of a fire. The problem is defined on an undirected finite graph $G = (V, E)$, where fire breaks out initially at f nodes. In each subsequent time-step, two actions occur: a certain number b of firefighters are placed on nonburning nodes, permanently protecting them from the fire, then the fire spreads to all nondefended neighbors of the nodes on fire. Because the graph is finite, at some point each node is either on fire or saved, and thus the fire cannot spread further. One of the objectives for the problem is to place the firefighters in such a way that the number of saved nodes is maximized.*

The applications of the Firefighter Problem reach from real fires to the spreading of diseases and the containment of floods. Furthermore, it can be used to model the spread of computer viruses or viral marketing in communication networks. Most research on the problem considers the case in which the fire starts in a single place (i.e., $f = 1$), and in which the budget of available firefighters per time-step is one (i.e., $b = 1$). So does the work in this study. This configuration already leads to hard problems and, even in this case, the problem is known to be NP-hard.

In this work, we study the problem from a game-theoretical perspective. We introduce a strategic game model for the Firefighter Problem to tackle its complexity from a different angle. We refer to it as the Firefighter Game. Such a game-based context seems very appropriate when applied to large networks where entities may act and make decisions based on their own interests, without global coordination.

At every time-step of the game, a player decides whether to place a new firefighter in a nonburning node of the graph. If so, he must decide where to place it. By placing it, the player is indirectly deciding which nodes to protect at that time-step. We define different utility functions in order to model selfish and nonselfish scenarios, which lead to equivalent games. We show that the Price of Anarchy (PoA) is linear for a particular family of graphs, but it is at most two for trees. We also analyze the quality of the equilibria when coalitions among players are allowed. It turns out that it is possible to compute an equilibrium in polynomial time, even for constant-size coalitions. This yields to a polynomial time approximation algorithm for the problem and its approximation ratio equals the PoA of the corresponding game. We show that for some specific topologies, the PoA is constant when constant-size coalitions are considered.

1. INTRODUCTION

The Firefighter Problem was introduced [25] as a deterministic discrete-time model for the spread and containment of fire. Since then, it has been subject to a wide variety

of research for modeling the spread and containment phenomena such as diseases, floods, ideas in social networks and viral marketing.

The Firefighter Problem is defined on an undirected finite graph $G = (V, E)$, where fire breaks out initially at f nodes. In each subsequent time-step, two actions occur: first, a certain number b of firefighters are placed on nonburning nodes, thus permanently protecting them from the fire; then the fire spreads to all nondefended neighbors of the nodes on fire. Because the graph is finite, at some point each node is either on fire or saved. Then the process finishes, because the fire cannot spread any further. There are several different objectives for the problem. Typically, the goal is to save the maximum number of nodes. Other objectives include minimizing the number of firefighters (or time-steps) until the spreading stops, or determining whether a specified collection of nodes can be prevented from burning.

Most research on the Firefighter Problem (including the work in this sudy) considers the case with only one starting place on fire and one available firefighter to protect a nonburning place at a time (i.e., $f = b = 1$), which already leads to hard problems. The problem was proved NP-hard for bipartite graphs [30], graphs with degree three [17], cubic graphs [28], and unit disk graphs [21]. However, the problem is polynomial-time solvable for various well-known graph classes, including interval graphs, split graphs, permutation graphs, caterpillars, and P_k-free graphs for fixed k [18, 23, 30, 21]. Furthermore, the problem is $(1 - 1/e)$-approximable on general trees [8], 1.3997-approximable for trees where nodes have at most three children [27], and it is NP-hard to approximate within $n^{(1-\varepsilon)}$ for any $\varepsilon > 0$ [2]. Later results on approximability for several variants of the problem can be found in [2, 4, 11].

Recently, the scientific community has focused on the study of the parameterized complexity of the problem. It was shown to be fixed parameter-tractable with respect to the combined parameter "pathwidth" and "maximum degree" [10]. Other important results can be found in [12, 3]. From this perspective, the problem is known to be fixed-parameter tractable (FPT) on trees in various parameterizations. When parameterized by the number of burned nodes, the problem is FPT on general graphs but has no polynomial kernel on trees. When parameterized by the number of unburned nodes, the problem is W[1]-hard even on bipartite graphs.

In 2000, the greedy algorithm for trees, which saves the vertex v that maximizes the number of nodes that will be saved if v is protected, was proved to be a $1/2$-approximation algorithm [26]. A linear programming relaxation for trees that supposedly gives a c-approximation algorithm was presented in 2006 [24], and a subexponential $(1 - 1/e)$-approximation method in 2008 [8]. These results have been improved in 2011 [27]. However, exact polynomial solutions exist for caterpillar and P-trees [18, 23, 30].

First approaches for grids of 2 and 3 dimensions were provided in 2002 [20, 34] and then generalized in 2007 [13]. These studies concluded that two firefighters were needed to contain the fire in an infinite 2-dimensional square grid and $2d - 1$ in a d-dimensional grid with $d \geq 3$. There exist concrete results for triangular, strong, and hexagonal grids [20, 31, 32, 33, 22], and for other graph classes [21].

The *surviving rate* of a graph is defined as the average percentage of nodes that can be saved when f fires break out at random nodes of the graph [7]. The study of this concept has become very fruitful in the literature and the evidence is the existence of many works on the subject for different graph structures [9, 6, 14, 16, 29, 36, 41, 44, 42, 43].

A complementary concept is called the *burning number* of a graph. It indicates how fast a graph can burn completely if there is a new fire in every round. It can be used as a measure of the speed of the spread of contagion in a graph [5].

For other variants of the Firefighter Problem see [15, 35, 18].

In this work, we introduce a game-theoretical model for the Firefighter Problem to tackle its complexity from a different angle. Our main objective is to try to bypass the NP-hardness of the problem of computing the optimal solution of the Firefighter Problem by taking into account different kinds of solutions. Our proposal as a possible set of solutions to the Firefighter Problem corresponds to the set of Nash equilibria of a new strategic game that we call the Firefighter Game. In this game there is a player for each time-step. Each player decides where to put the firefighters at his corresponding time-step. The goal of each player is to save as many nodes as possible. The outcomes of this strategic game represent solutions to the classical Firefighter Problem. Game theory provides a powerful toolset for analyzing the quality of stable outcomes or equilibria.

To the best of our knowledge, the only existing game-theoretical models to similar problems are those referred to as the vaccination problem [2, 19], the spreading of rumors [45, 46], and competitive diffusion [1, 37, 40, 39, 38]. Those models, however, focus on information spreading on social networks, and, thus, take into account other inherent aspects of those scenarios, such as preferences, reputation, popularity, and other personal traits of the users, and relevance or truthfulness of the information. Our proposal is well suited to model fighting against spreading phenomena in large networks, where the protection strategy for each time-step is decided by one player, independently from the others. We believe that our model has legitimacy, especially if this problem is used to model processes that take place on some modern networks where no central entity exists. In this case, the resources to fight the spreading have to be delegated by independent agents, who might not be able to compute a global solution. Once we have defined formally the model, we focus on the study of the Nash equilibria and the Price of Anarchy as measures for the quality of the equilibria.

This article is organized as follows. In Section 2 we define some basic game-theoretical concepts extensively used throughout, most importantly the measures of the quality of equilibria, i.e., the Price of Anarchy. In Section 3 we introduce the game and we study the set of equilibria corresponding to two different utility functions. In Section 4 we analyze the Price of Anarchy for different families of graphs. In Section 5 we consider that the players might form coalitions and we analyze the equilibria for constant size coalitions. Finally, conclusions and directions for future work can be found in Section 7.

2. GAME-THEORETICAL DEFINITIONS

Let us introduce some basic definitions. A *strategic game* $\mathcal{G}$ is defined by a tuple $\mathcal{G} = (\mathcal{N}, (\mathcal{S}_i)_{i \in \mathcal{N}}, (u_i)_{i \in \mathcal{N}})$, where

- $\mathcal{N}$ denotes the set of players,
- $\mathcal{S}_i$ is the set of *actions* of player $i \in \mathcal{N}$, and
- u_i is the *utility function* of player $i \in \mathcal{N}$.

The *strategy* s_i of a player i is to select an action $s_i \in \mathcal{S}_i$. Let $s = (s_1, \ldots, s_{|\mathcal{N}|})$ be a *strategy profile* describing the strategies s_i of each player $i \in \mathcal{N}$. We denote by $\mathcal{S}$ the set of all strategy profiles, $\mathcal{S} = \mathcal{S}_1 \times \ldots \times \mathcal{S}_{|\mathcal{N}|}$. The utility function of player i, $u_i : \mathcal{S} \to \mathbb{R}$, assigns to each strategy profile $(s_1, \ldots, s_{|\mathcal{N}|}) \in \mathcal{S}$ a value $u_i(s_1, \ldots, s_{|\mathcal{N}|})$. This value quantifies the benefit that player i receives by selecting strategy s_i when the strategies of other players are $(s_j)_{j \in \mathcal{N}, j \neq i}$.

Given a strategy profile $s = (s_1, \ldots, s_{|\mathcal{N}|})$, we denote $s_{-i} = (s_j)_{j \in \mathcal{N} \setminus \{i\}}$. Furthermore, we denote $(s_{-i}, s_i') = (s_1, \ldots, s_i', \ldots, s_{|\mathcal{N}|})$, i.e., strategy vector s, where player i changed

his strategy from s_i to s_i'. The social benefit of the outcome of the game for any strategy profile is quantified by a function that is called *Social Welfare* $W : \mathcal{S} \to \mathbb{R}$.

Definition 2.1. (Nash Equilibrium.) A strategy profile of a game is a Nash equilibrium if no player can improve his payoff by changing his strategy. Formally, given a strategic game $\mathcal{G} = (\mathcal{N}, (\mathcal{S}_i)_{i \in \mathcal{N}}, (u_i)_{i \in \mathcal{N}})$, a strategy profile s is a Nash equilibrium of $\mathcal{G}$ if and only if

$$\forall i \in \mathcal{N}, \forall s_i' \in \mathcal{S}_i : u_i(s) \geq u_i(s_{-i}, s_i').$$

We denote by $\text{NE}(\mathcal{G})$ the set of all Nash equilibria of $\mathcal{G}$. Whenever there is no possible confusion, we will use the notation NE without any reference to the game. In order to measure the quality of the equilibria, Koutsoupias and Papadimitriou introduced the *Price of Anarchy* concept in 1999.

Definition 2.2. (Price of Anarchy.) The Price of Anarchy (PoA) of a game $\mathcal{G}$ (with respect to a social welfare function W) is defined as follows:

$$\text{PoA}(\mathcal{G}) = \frac{\max_{s \in \mathcal{S}} W(s)}{\min_{s \in \text{NE}} W(s)}.$$

If, instead of considering the worst equilibria with relation to the social welfare, we consider the best equilibria, then we have the *Price of Stability*.

Definition 2.3. (Price of Stability.) The Price of Stability (PoS) of a game $\mathcal{G}$ (with respect to a social welfare function W) is defined as follows:

$$\text{PoS}(\mathcal{G}) = \frac{\max_{s \in \mathcal{S}} W(s)}{\max_{s \in \text{NE}} W(s)}.$$

3. THE FIREFIGHTING GAME

Each instance of the Firefighting Problem is defined by an undirected graph $G = (V, E)$ and a marked node $v_0 \in V$. We assume that, initially, the fire breaks out at v_0 and it burns. At each time-step $t \geq 1$ the burned nodes at time step $t - 1$ incinerate all their neighboring nodes. Let us call them *burning* nodes. At each time-step a fixed number of firefighters b, called *budget*, can be placed on nonburning nodes to permanently protect them from burning. These nodes are called *defended*. If a node never burns because it is defended or cut off from the fire it is called *saved*. All other nodes are called *vulnerable*. We consider only the case when the budget is of size one (i.e., $b = 1$).

3.1. Players and Strategies

In order to model the Firefighter problem as a game, we define a set of players $\mathcal{N} = \{1, \ldots, n - 1\}$ where $n = |V|$. Player i represents the firefighter who selects the set of nodes to be protected at time step i. Hence, the set of all possible strategies of player i is defined by $\mathcal{S}_i = \{U \mid U \subseteq V\}$. The strategy of player i consists of selecting a subset of nodes $U \in \mathcal{S}_i$. Because we deal only with the case of $b = 1$, we overload notation,

and instead of subsets of size one, we set the strategies to the vertices themselves or to the empty set, i.e., $\mathcal{S}_i = V \cup \{\emptyset\}$. This means that every player can choose one node or the empty set as strategy. Note that player $i \in \mathcal{N}$ applies its strategy s_i at time-step i. Let $s = (s_1, \ldots, s_{|\mathcal{N}|})$ denote the strategy profile of all players.

3.2. The Outcome of a Strategy Profile

The outcome of a strategy profile $s = (s_1, \ldots, s_{|\mathcal{N}|})$ on a given graph $G = (V, E)$ is a partition of V into saved and burning nodes. It is defined in the following way. At time-step 0, the only burning node is v_0. At time-step $i > 0$, two events occur: first, player i's node is protected if his action is valid w.r.t. to strategy profile s, i.e., it is neither burning nor already defended at the end of time-step $i - 1$. Second, each node burning at time-step $i - 1$ incinerates all its nondefended neighbors. The process stops when the fire cannot spread any further. Let us denote by Safe$(s) \subset V$ the set of nodes that are saved by the strategy profile s.

In the following we introduce some predicates in order to define formally the outcome of a strategy profile on a given graph G.

Notice that, when no further restrictions apply on the strategies for each player, then it might be the case that a player selects a node that is already burning or that already has a firefighter on it. In order to identify these situations, we introduce additional predicates. Given a strategy profile s, a node $v \in V$, and a player $i \in \mathcal{N}$, we define

- defended$(s, v, i) = true$ iff in the strategy profile s, node v is defended at the end of time-step i,
- burning$(s, v, i) = true$ iff in the strategy profile s, node v is burning at the end of time-step i, and
- invalid$(s, i) = true$ iff player i's action is not valid with respect to strategy vector s, i.e., player i wants to place a firefighter either on a node that already is defended at time-step $i - 1$ or on a node that is burning at time-step $i - 1$.

Let Safe$_i(s)$ denote the set of nodes that would burn if player i switched his action to the empty set. Formally, Safe$_i(s) = $ Safe$(s) \setminus $ Safe$(s_{-i}, \emptyset)$.

At this point we can define all the previous predicates inductively on $i \geq 0$. For the case $i = 0$, we have no player, but the process starts by burning v_0. Formally:

$$\forall s \in \mathcal{S}, \forall v \in V : \text{defended}(s, v, 0) = false. \tag{3.1}$$

$$\forall s \in \mathcal{S}, \forall v \in V \setminus \{v_0\} : \text{burning}(s, v, 0) = false. \tag{3.2}$$

$$\forall s \in \mathcal{S} : \text{burning}(s, v_0, 0) = true. \tag{3.3}$$

These equations define the starting configuration. Equation (3.1) makes sure that all nodes are undefended at time 0. Equation (3.3) defines that the fire starts at node v_0, and (3.2) declares all other nodes as nonburning.

For any player $i \in \mathcal{N}$ (or time-step $i \geq 1$) and for any $s \in \mathcal{S}$ we define as follows

$$\text{invalid}(s, i) = \begin{cases} \text{false} & \text{if } s_i = \emptyset, \\ \bigvee_{v \in s_i} \Big(\text{defended}(s, v, i-1) \vee \text{burning}(s, v, i-1) \Big) & \text{otherwise.} \end{cases} \tag{3.4}$$

The first case in (3.4) makes sure that it is always valid to play the empty set. The second case states that the move of player i is invalid if he tries to put a firefighter on a node that is either already defended or already burning. We can also define inductively on $i \geq 1$ the predicates defended and burning.

$$\forall s \in \mathcal{S}, \forall v \in V : \text{defended}(s, v, i) = \bigvee_{j \in \mathcal{N} \wedge j \leq i} \Big(v \in s_j \wedge \neg\text{invalid}(s, j)\Big). \quad (3.5)$$

Equation (3.5) states that a node v is defended at (the end of) time-step i if one of the players who has made a valid move until then has decided to put a firefighter on node v.

To define that a node v is burning at (the end of) time-step $i \geq 1$, we need the neighborhood relation of nodes. Let $N(v)$ denote the neighborhood of v, i.e., $w \in N(v)$ if, and only if, $(v, w) \in E$. Then,

$$\forall s \in \mathcal{S}, v \in V, i \geq 1 :$$
$$\text{burning}(s, v, i) = \neg\text{defended}(s, v, i) \wedge \bigvee_{w \in N(v)} \Big(\text{burning}(s, w, i - 1)\Big). \quad (3.6)$$

Equation (3.6) states that a node v is burning at time-step i if it is not defended by the end of time-step i and at least one of its neighbors is burning at time-step $i - 1$. We can also define the set of all nodes that will be saved when a strategy vector s is played, which are those that are still not burning at time-step $n - 1$.

$$\forall s \in \mathcal{S} : \text{Safe}(s) = \{v \in V \mid \neg\text{burning}(s, v, n - 1)\}.$$

Furthermore, we can define the set of nodes that a player i helped to save, i.e., that would burn if player i changed his strategy to the empty set.

$$\forall s \in \mathcal{S}, \forall i \in \mathcal{N} : \text{Safe}_i(s) = \text{Safe}(s) \setminus \text{Safe}(s_{-i}, \emptyset).$$

We can consider two different and natural utility functions. The first models a selfish behavior, whereas the second models a nonprofitable behavior.

3.3. Selfish Firefighters Model

In this model, firefighters get paid for the nodes they save. Intuitively, if player i makes a valid move other than the empty set, he gets one unit of currency from each node he helped to save. In other words, he gets paid by all nodes that are saved with respect to the played strategy vector, but would not be saved if he was to change his strategy to the empty set. Additionally, he will get charged a penalty if he makes an invalid move. Now, let us define the utility function formally.

$$u_i^{(\text{Selfish})}(s) = \begin{cases} -c & \text{if invalid}(s, i), \\ 0 & \text{if } s_i = \emptyset, \\ |\text{Safe}_i(s)| - \varepsilon & \text{otherwise}, \end{cases}$$

with $0 < \varepsilon < 1$ and $c > 0$. We can see that the definition follows the intuition very closely. Subtracting an ε cost for placing a firefighter makes sure that players always prefer to play

the empty set over placing a firefighter on a node that is already saved (which would not be an invalid move).

Hence, a game $\mathcal{G}^{(\text{Selfish})}$ in this Selfish Firefighters Model is defined by a tuple $\mathcal{G}^{(\text{Selfish})} = (G, v_0, \varepsilon, c)$, where $G = (V, E)$ is an undirected graph, $v_0 \in V$ is the initial burning node, and ε, c are constants. The utility function of each player $i \in \mathcal{N}$ is defined as $u_i^{(\text{Selfish})}$.

3.4. Nonprofit Firefighters Model

Here we assume that the goal of every firefighter is to save as many total nodes as possible, independently of which firefighters actually save more nodes. Formally, we define

$$u_i^{(\text{Nonprofit})}(s) = \begin{cases} -c & \text{if invalid}(s, i), \\ |\text{Safe}(s)| & \text{if } s_i = \emptyset, \\ |\text{Safe}(s)| - \varepsilon & \text{otherwise}, \end{cases}$$

with $0 < \varepsilon < 1$ and $c > 0$. Hence, a game $\mathcal{G}^{(\text{Nonprofit})}$ in this Nonprofit Firefighters Model is defined by a tuple $\mathcal{G}^{(\text{Nonprofit})} = (G, v_0, \varepsilon, c)$, where $G = (V, E)$ is an undirected graph, $v_0 \in V$ is the initial burning node, and ε, c are constants (w.r.t $n = |V|$). The utility function of each player $i \in \mathcal{N}$ is defined as $u_i^{(\text{Nonprofit})}$. Let us denote by $\mathcal{G}^{(\text{Selfish})}(G, v_0, \varepsilon, c)$ and by $\mathcal{G}^{(\text{Nonprofit})}(G, v_0, \varepsilon, c)$ the games defined by G, v_0, ε, and c in the models Selfish Firefighters and Nonprofit Firefighters, respectively. For simplicity, we also denote by $\text{NE}(\mathcal{G})$ the set of NE of the game $\mathcal{G}$.

Notice that in an equilibrium, no player plays an invalid move or puts a firefighter on a safe node. Also, since we have that $0 < \varepsilon < 1$, the utility of playing the empty set is less than the utility of saving one node. Because of that, given that a player does not play the empty set, the ε-value does not affect his preferences. Therefore, we will ignore it in the proofs.

Proposition 3.1. *For any graph $G = (V, E)$, any initial node $v_0 \in V$, and any constants $\varepsilon, \varepsilon', c, c'$ such that $0 < \varepsilon, \varepsilon' < 1$ and $c, c' > 0$, we have that:*

i) $\text{NE}(\mathcal{G}^{(\text{Selfish})}(G, v_0, \varepsilon, c)) = \text{NE}(\mathcal{G}^{(\text{Selfish})}(G, v_0, \varepsilon', c'))$,
ii) $\text{NE}(\mathcal{G}^{(\text{Nonprofit})}(G, v_0, \varepsilon, c)) = \text{NE}(\mathcal{G}^{(\text{Nonprofit})}(G, v_0, \varepsilon', c'))$.

From now on we will omit ε and c in the definition of the games.

3.5. Equivalence of Games

Surprisingly, the behavior of selfish firefighters leads to the same equilibria as the behavior of the nonprofit firefighters.

Let us now formally show that the games $\mathcal{G}^{(\text{Selfish})}$ and $\mathcal{G}^{(\text{Nonprofit})}$ are equivalent in the sense that their sets of equilibria are the same.

Proposition 3.2. *For any graph $G = (V, E)$ and any $v_0 \in V$, if*

$$\mathcal{G}^{(\text{Selfish})} = \mathcal{G}^{(\text{Selfish})}(G, v_0), \text{ and } \mathcal{G}^{(\text{Nonprofit})} = \mathcal{G}^{(\text{Nonprofit})}(G, v_0), \text{ then}$$

$$\text{NE}(\mathcal{G}^{(\text{Selfish})}) = \text{NE}(\mathcal{G}^{(\text{Nonprofit})}).$$

Proof.

(i) Let us show that $\text{NE}(\mathcal{G}^{(\text{Selfish})}) \subseteq \text{NE}(\mathcal{G}^{(\text{Nonprofit})})$:

Let $s = (s_1, ..., s_n) \in \text{NE}(\mathcal{G}^{(\text{Selfish})})$. If $s \notin \text{NE}(\mathcal{G}^{(\text{Nonprofit})})$, then there is a player i who can improve his payoff by changing his strategy from s_i to another s_i'. If we denote by $s' = (s_{-i}, s_i')$, we have that $u_i^{(\text{Nonprofit})}(s) < u_i^{(\text{Nonprofit})}(s')$, and then $|\text{Safe}(s)| < |\text{Safe}(s')|$.

By definition,

$$\text{Safe}(s) = \text{Safe}(s_{-i}, \emptyset) \cup \text{Safe}_i(s) \text{ , and}$$

$$\text{Safe}(s') = \text{Safe}(s_{-i}, \emptyset) \cup \text{Safe}_i(s').$$

Notice that $\text{Safe}(s_{-i}, \emptyset) = \text{Safe}(s_{-i}', \emptyset)$ and $\text{Safe}_i(s') \cap \text{Safe}(s_{-i}, \emptyset) = \emptyset$. Then, the cardinalities of $\text{Safe}(s')$ and $\text{Safe}(s')$ can be expressed as:

$$|\text{Safe}(s')| = |\text{Safe}(s_{-i}, \emptyset)| + |\text{Safe}_i(s')|, \tag{3.7}$$

$$|\text{Safe}(s)| = |\text{Safe}(s_{-i}, \emptyset)| + |\text{Safe}_i(s)|. \tag{3.8}$$

Because we have assumed that $|\text{Safe}(s)| < |\text{Safe}(s')|$, then $|\text{Safe}_i(s)| < |\text{Safe}_i(s')|$, contradicting the fact that $s \in \text{NE}(\mathcal{G}^{(\text{Selfish})})$.

(ii) Let us now prove that $\text{NE}(\mathcal{G}^{(\text{Nonprofit})}) \subseteq \text{NE}(\mathcal{G}^{(\text{Selfish})})$:

Let us suppose that there is a strategy $s = (s_1, ..., s_n) \in \text{NE}(\mathcal{G}^{(\text{Nonprofit})})$, which is not in $\text{NE}(\mathcal{G}^{(\text{Selfish})})$. Then, there exists a player i and a strategy $s_i' \neq s_i$ such that $|\text{Safe}_i(s)| < |\text{Safe}_i(s')|$ where $s' = (s_{-i}, s_i)$.

Since we can express the cardinalities as in (3.7) and (3.8), and $|\text{Safe}_i(s)| < |\text{Safe}_i(s')|$, then

$$|\text{Safe}(s)| < |\text{Safe}(s')|.$$

But this is a contraction with the fact that

$$s = (s_1, ..., s_n) \in \text{NE}(\mathcal{G}^{(\text{Nonprofit})}).$$

$\square$

Social welfare. We define the social welfare of a strategy as the number of nodes that are saved, i.e., $W(s) = |\text{Safe}(s)|$.

Corollary 3.3. *Let $G = (V, E)$ be a graph and let $v_0 \in V$. Let $\mathcal{G}^{(\text{Selfish})} = \mathcal{G}^{(\text{Selfish})}(G, v_0)$ and $\mathcal{G}^{(\text{Nonprofit})} = \mathcal{G}^{(\text{Nonprofit})}(G, v_0)$. Then,*

$$\text{PoA}(\mathcal{G}^{(\text{Selfish})}) = \text{PoA}(\mathcal{G}^{(\text{Nonprofit})}),$$

$$\text{PoS}(\mathcal{G}^{(\text{Selfish})}) = \text{PoS}(\mathcal{G}^{(\text{Nonprofit})}).$$

Therefore, we will use the utility function, which is more convenient for the proofs. From now on, we will also refer to a *firefighter game* with $\mathcal{G}$, whenever the respective result holds for both the selfish and the nonprofit model.
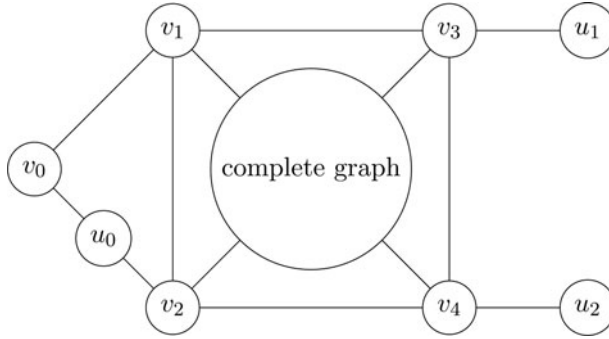
Figure 1 Family of graphs $G_{PoA}(n) = (V_{PoA}(n), E_{PoA}(n))$.

4. QUALITY OF EQUILIBRIA

Once we have established the model, we can analyze the quality of the equilibria. It is easy to argue that equilibria always exist. Notice that in the Nonprofit Firefighter Model, the social welfare of a strategy s coincides with the utility of every player i on such strategy, $W(s) = |\text{Safe}(s)| = u_i^{(\text{Nonprofit})}(s)$. Hence, in the case of nonprofit firefighters, every strategy that maximizes the social welfare also maximizes the utility of every player i. In an optimal solution s no player protects nodes that are already saved or already burned. Then s not only is an optimal solution but also a Nash equilibrium. Therefore, we have the PoS is 1. This is independent of the class of graphs we are considering and holds for every solution concept by which players maximize their utility function.

Proposition 4.1. *For any firefighter game $\mathcal{G}$,*

$$\text{PoS}(\mathcal{G}) = 1.$$

In contrast to the PoS, the PoA might be very high in this model. In Figure 1 we depict a family of graphs $G_{PoA}(n) = (V_{PoA}(n), E_{PoA}(n))$ that allows us to show a linear lower bound for the PoA. Note that $(v_1, v_4) \in E_{PoA}(n)$ and $(v_2, v_3) \in E_{PoA}(n)$. For better visibility, these edges are not drawn in the picture. Further, we have that $|V_{PoA}(n)| = n$, hence, the size of the complete subgraph is $n - 8$, and nodes v_1, v_2, v_3, and v_4 are connected to every node in the complete subgraph.

Theorem 4.2. *For any $n \geq 1$, the firefighter game $\mathcal{G} = \mathcal{G}(G_{PoA}(n), v_0)$ satisfies that*

$$\text{PoA}(\mathcal{G}) = \Theta(n).$$

Proof. Notice that any strategy profile s of the game $\mathcal{G}$ satisfies that $\text{Safe}(s) \leq n$. Hence, we need to prove only that $\text{PoA}(\mathcal{G}) = \Omega(n)$.

Recall that the fire starts at v_0. It is easy to see that $s = (\{v_1\}, \{v_2\}, \emptyset^{n-3})$ is the optimal strategy. Only nodes v_0 and u_0 burn, hence, the social welfare is $W(s) = n - 2$. Furthermore, we have that $s' = (\{v_3\}, \{v_4\}, \emptyset^{n-3})$ is an equilibrium. Note that the complete graph is burning after two time-steps, therefore, at time-step 3 only u_1 and u_2 are neither burning nor defended. But these nodes are already safe, hence, players i with $i > 2$ will

not place firefighters on them. Furthermore, players 1 and 2 cannot improve their payoff, because if one of them changes strategy, that player will save at most one node. The social welfare of s' is $W(s') = 4$. Hence, we have that $\text{PoA}(\mathcal{G}) \geq \frac{n-2}{4}$. It follows that $\text{PoA}(\mathcal{G}) \in \Omega(n)$. $\square$

The Price of Anarchy for trees. Because the PoA might be very high in general, we are interested in studying the quality of equilibria for particular topologies. Our aim is to prove that there are cases where the quality of the equilibria is close to the quality of an optimal solution. In the following we analyze the PoA on tree topologies.

Theorem 4.3. *For any tree $T = (V, E)$ and any node $v_0 \in V$, the firefighter game $\mathcal{G}_{Tree} = \mathcal{G}(T, v_0)$ satisfies that:*

$$PoA(\mathcal{G}_{Tree}) \leq 2.$$

Proof. In this proof, we use ideas similar to the proof of the approximation ratio of a greedy algorithm [26]. We assume that the initial burning node v_0 is the root of the tree. We use the utility functions of the selfish firefighters. This implies that the utility $u_i^{(Selfish)}$ of player i is less than, or equal to, the size of the subtree rooted by s_i.

Let $s = (s_1, \ldots, s_{|\mathcal{N}|})$ be a NE, and let $\text{opt} = (\text{opt}_1, \ldots, \text{opt}_{|\mathcal{N}|})$ be an optimal solution. Notice that for each pair of different players i, j, we have that s_i cannot be an ancestor of s_j. Even though it can be the case that opt_i is an ancestor of opt_j, we can always consider an equivalent strategy $\text{opt}' = (\text{opt}_{-j}, \emptyset)$ such that $W(\text{opt}') = W(\text{opt})$. Hence, there exists an optimal strategy opt' such that, for every pair $i, j, i \neq j$, the subtrees rooted at opt'_i and opt'_j, respectively, are disjoint.

Let opt_A be the set of optimal actions opt_i that they are neither equal to any strategy s_j nor successors of any strategy s_j. Formally,

$$\text{opt}_A = \{\text{opt}_i \mid \forall j \in \mathcal{N} : s_j \neq \text{opt}_i \wedge s_j \text{ is not an ancestor of } \text{opt}_i\}.$$

Let opt_B denote the remaining optimal actions:

$$\text{opt}_B = \{\text{opt}_j \mid j \in \mathcal{N} \wedge \text{opt}_j \notin \text{opt}_A\}.$$

Let $P(\text{opt}_i)$ denote the set of strategies s_j that are successors of opt_i. Let s_A denote the set of strategies, that do not have an optimal action as an ancestor. Formally,

$$s_A = \{s_i \mid \forall j \in \mathcal{N} : \text{opt}_j \text{ is not an ancestor of } s_i\}.$$

Let s_B denote the remaining player actions.

$$s_B = \{\text{s}_j \mid j \in \mathcal{N} \wedge s_j \notin s_A\}.$$

Let $\text{save}(a)$ denote the number of nodes saved by action a. Notice that $\text{save}(s_i) = u_i(s)$ (the size of the subtree rooted at s_i) and $\text{save}(\text{opt}_j)$ is equal to the size of the subtree rooted at opt_j. Moreover, because the subtrees rooted at opt_i and opt_j are disjoint, then $W(\text{opt}) = \sum_{i \in \mathcal{N}} \text{save}(\text{opt}_i)$. Therefore, we have that

$$\sum_{i \in \mathcal{N} \,|\, \mathrm{opt}_i \in \mathrm{opt}_B} \mathrm{save}(\mathrm{opt}_i) \leq \sum_{i \in \mathcal{N} \,|\, s_i \in s_A} \mathrm{save}(s_i). \tag{4.1}$$

Because s is a NE, then for each $i \in \mathcal{N}$

$$\mathrm{save}(s_i) \geq \mathrm{save}(\mathrm{opt}_i) - \sum_{s_j \in P(\mathrm{opt}_i)} \mathrm{save}(s_j),$$

otherwise player i would have an incentive to switch his strategy to opt_i. Summing up all optimal actions in opt_A, we get

$$\sum_{i \in \mathcal{N} \,|\, \mathrm{opt}_i \in \mathrm{opt}_A} \mathrm{save}(\mathrm{opt}_i) \leq \sum_{i \in \mathcal{N} \,|\, \mathrm{opt}_i \in \mathrm{opt}_A} \left(\mathrm{save}(s_i) + \sum_{s_j \in P(\mathrm{opt}_i)} \mathrm{save}(s_j) \right).$$

We can split the sum on the right-hand side and get

$$\sum_{i \in \mathcal{N} \,|\, \mathrm{opt}_i \in \mathrm{opt}_A} \mathrm{save}(s_i) + \sum_{i \in \mathcal{N} \,|\, \mathrm{opt}_i \in \mathrm{opt}_A} \sum_{s_j \in P(\mathrm{opt}_i)} \mathrm{save}(s_j).$$

Note that in the double sum, we sum up exactly over the player actions that have an optimal action as an ancestor i.e., s_B. So, we can rewrite this to

$$\sum_{i \in \mathcal{N} \,|\, \mathrm{opt}_i \in \mathrm{opt}_A} \mathrm{save}(\mathrm{opt}_i) \leq \sum_{i \in \mathcal{N} \,|\, \mathrm{opt}_i \in \mathrm{opt}_A} \mathrm{save}(s_i) + \sum_{i \in \mathcal{N} \,|\, s_i \in s_B} \mathrm{save}(s_i).$$

Now we can use 4.1 to get

$$\sum_{i \in \mathcal{N}} \mathrm{save}(\mathrm{opt}_i) \leq \sum_{i \in \mathcal{N} \,|\, \mathrm{opt}_i \in \mathrm{opt}_A} \mathrm{save}(s_i) + \sum_{i \in \mathcal{N}} \mathrm{save}(s_i).$$

Furthermore, we have that $\sum_{\mathrm{opt}_i \in \mathrm{opt}_A} \mathrm{save}(s_i) \leq \sum_{s_i \in s} \mathrm{save}(s_i)$, which yields to

$$\sum_{i \in \mathcal{N}} \mathrm{save}(\mathrm{opt}_i) \leq 2 \sum_{i \in \mathcal{N}} \mathrm{save}(s_i).$$

Since $W(s) = \sum_{i \in \mathcal{N}} \mathrm{save}(s_i)$ and $W(\mathrm{opt}) = \sum_{i \in \mathcal{N}} \mathrm{save}(\mathrm{opt}_i)$, we have that $\mathrm{PoA}(\mathcal{G}_{\mathrm{tree}}) \leq 2$. $\qquad\square$

As we will see in Section 6, the problem of finding a NE of a given firefighter game $\mathcal{G}$ is polynomial time computable. Hence, the algorithm of computing a NE yields a polynomial time approximation algorithm for the firefighter problem with an approximation ratio bounded by PoA. In the case of the trees, the algorithm of computing a NE provides an approximation algorithm with an approximation ratio upper bounded by 2 (the number of nodes saved by an optimum solution is at most twice the number of nodes saved by a NE solution as in the case of the greedy algorithm presented in [26]).
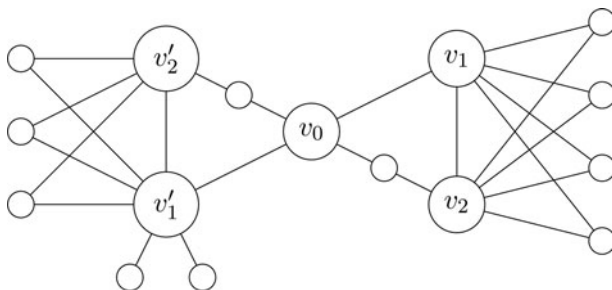
Figure 2 Example graph. Note that all nodes have distance at most two from the fire.

5. COALITIONS

In this section we consider that players may form coalitions among themselves. A coalition is willing to deviate from the strategy as long as no player in the coalition loses payoff and at least one player increases his utility. We show that this might affect the PoA. We consider that a strategy vector s is an equilibrium strategy with respect to coalition size k, if no set of at most k players can simultaneously change their strategies in such a way that at least one player increases his payoff and no player of the coalition decreases his payoff. In the following we define formally these concepts.

Let $K \subseteq \mathcal{N}$ be a coalition set and let s be a strategy profile of all the players in $\mathcal{N}$. Let s_K denote the joint strategy profile of all the players in K, $s_K = (s_i)_{i \in K}$. We say that coalition K has an *attractive joint deviation* from s when there exists a joint strategy profile s'_K, such that for all $i \in K$, $u_i(s) \leq u_i(s_{-K}, s'_K)$ and there is at least one player $j \in K$ such that $u_j(s) < u_j(s_{-K}, s'_K)$.

We say that a strategy profile s is an *equilibrium with respect to coalition size k* if for every $K \subseteq \mathcal{N}$ such that $|K| \leq k$, coalition K does not have any attractive deviation from s. Let $E_k \subseteq \mathcal{S}$ denote the set of all equilibria with respect to coalition size k.

In the case of equilibria with respect to a coalition size, we do not have an equivalence between selfish and non profit firefighters as in the Nash case.

Lemma 5.1. *There exists a graph G, a node $v_0 \in V(G)$, and a coalition size $k > 1$ such that the firefighter games $\mathcal{G}^{(Selfish)} = \mathcal{G}^{(Selfish)}(G, v_0)$ and $\mathcal{G}^{(nonprofit)} = \mathcal{G}^{(nonprofit)}(G, v_0)$ satisfy that*

$$E_k(\mathcal{G}^{(Selfish)}) \nsubseteq E_k(\mathcal{G}^{(Nonprofit)}).$$

Proof. Consider the graph in Figure 2 and assume a coalition size $k \geq 2$. Note that only Players 1 and 2 can make meaningful moves, hence, without loss of generality we denote the strategy vector with $s = (s_1, s_2)$.

We have that $s = (\{v_1\}, \{v_2\})$ is an equilibrium strategy for the selfish firefighters, because deviating to $s' = (\{v'_1\}, \{v'_2\})$ would decrease the utility of Player 2. However, s is not an equilibrium strategy for nonprofit firefighters, because the joint deviation s' increases the total number of saved nodes. □

Lemma 5.2. *There exists a graph G, a node $v_0 \in V(G)$, and a coalition size $k > 1$ such that the firefighter games $\mathcal{G}^{(Selfish)} = \mathcal{G}^{(Selfish)}(G, v_0)$ and $\mathcal{G}^{(Nonprofit)} = \mathcal{G}^{(Nonprofit)}(G, v_0)$*

satisfy that

$$E_k(\mathcal{G}^{(Nonprofit)}) \nsubseteq E_k(\mathcal{G}^{(Selfish)}).$$

Proof. Consider the graph in Figure 3 and assume a coalition size $k \geq 2$. Note that, at most, the first 3 players can make meaningful moves, hence, without loss of generality, we denote the strategy vector with $s = (s_1, s_2, s_3)$.
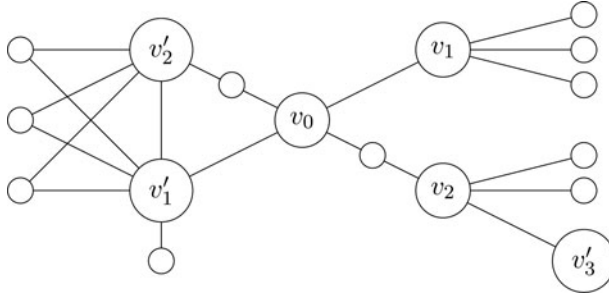

Figure 3 Example graph. Note that all nodes have distance at most three from the fire.

We have that $s = (\{v_1\}, \{v_2\}, \emptyset)$ is an equilibrium strategy for the nonprofit firefighters, because deviating to $s' = (\{v_1'\}, \{v_2'\}, \{v_3'\})$ would decrease the total number of saved nodes. However, s is not an equilibrium strategy for selfish firefighters, because the joint deviation $s_K' = (\{v_1'\}, \{v_2'\})$ strictly increases the utility of player 1 without decreasing that of player 2. $\square$

From now on we will consider only nonprofit firefighters because they resemble the usual objective to save as many nodes as possible.

5.1. Price of Anarchy

We analyze the PoA for coalitions and its relation with the coalition size.

Definition 5.3. (Price of Anarchy for coalitions.) The PoA for a coalition size k (PoA_k) of a game $\mathcal{G}$ is defined as follows:

$$\text{PoA}_k(\mathcal{G}) = \frac{\max_{s \in \mathcal{S}} W(s)}{\min_{s \in E_k} W(s)}.$$

We show that the PoA for a coalition size k is $O(\frac{n}{k})$ for any class of graphs.

Proposition 5.4. *Let $k > 1$ be the coalition size. Any nonprofit firefighter game $\mathcal{G} = \mathcal{G}(G, v_0)$ where $|V(G)| = n \geq 2k$ and any $v_0 \in V(G)$ satisfies that*

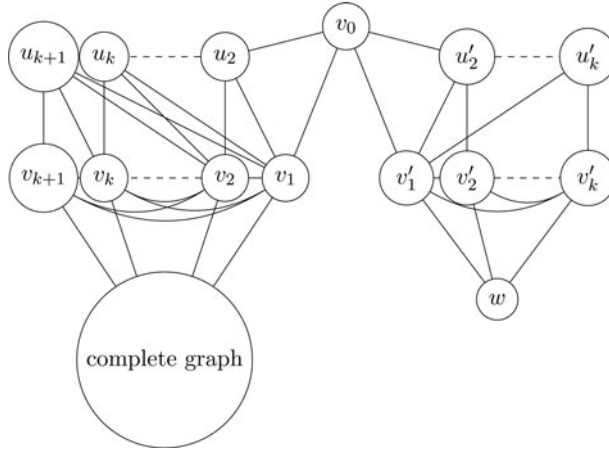$$\text{PoA}_k(\mathcal{G}) \leq frac{n}{k} - 1.$$

Figure 4 Family of graphs $G_{PoA}(n, k) = (V_{PoA}(n, k), E_{PoA}(n, k))$, with $|V_{PoA}(n, k)| = n$.

Proof. In order to show this bound, we bound the welfare of the optimal solution from above and bound the welfare of the worst equilibrium from below. Let us consider the following two cases:

(i) If there is an optimal solution that uses k or fewer firefighters, a coalition of size k can always make a joint deviation to that solution. Hence, there are no equilibria that have a lower welfare than the optimal solution. It follows that $\text{PoA}_k(\mathcal{G}) = 1$ for those instances.

(ii) If every optimal solution uses strictly more than k firefighters, the optimal solution saves at most $n - (k + 1)$ nodes. This is because it uses at least $k + 1$ time-steps and at least one node burns every time-step, otherwise the fire would be contained. An equilibrium, however, always saves at least k nodes in this case, because for every strategy profile in which the players save fewer than k nodes, the first k players can jointly deviate to the first k steps of the optimal solution, saving at least the nodes they protect, i.e., k. This yields the desired bound: $\text{PoA}_k(\mathcal{G}) \leq \frac{n}{k} - 1$.

$\square$

Furthermore, we can also show that there exists a family of graphs in which this upper bound is tight. Figure 4 shows the construction of the family of graphs $G_{PoA}(n, k) = (V_{PoA}(n, k), E_{PoA}(n, k))$, with $|V_{PoA}(n, k)| = n$. All nodes that are not specifically depicted are inside the complete subgraph. Hence, the size of the complete subgraph is $n - 4k - 2$. Note that nodes v_1 to v_{k+1} are connected to every node in the complete subgraph. The nodes v'_1 to v'_k together with w form a clique. Furthermore, for every v_i and u_j and for every v'_i and u'_j, there are edges (v_i, u_j) and (v'_i, u'_j), respectively, if $i \leq j$. Last, for every u_i and u'_i, there is an edge to u_{i+1} and u'_{i+1}, respectively. We refer to v_0 as the initial fire, nodes v_1 to v_{k+1}, u_2 to u_{k+1} and the complete subgraph as the left part of the graph, and the rest as the right part of the graph.

Theorem 5.5. *Let $k > 1$ be the coalition size. For any $n \geq 2k$, the nonprofit firefighter game $\mathcal{G} = (G_{PoA}(n, k), v_0)$ satisfies that*

$$\text{PoA}_k(\mathcal{G}) = \Theta\left(\frac{n}{k}\right).$$

Proof. Let us show that $\text{PoA}_k(\mathcal{G}) = \Omega(\frac{n}{k})$. Notice that the welfare of any strategy profile is a lower bound for the welfare of an optimal solution, whereas the welfare of any equilibrium gives an upper bound for the worst equilibrium in terms of quality.

The strategy profile $s^* = (v_1, v_2, \ldots, v_{k+1}, \emptyset^{|\mathcal{N}|-k-1})$ saves all but $3k+1$ nodes. The nodes that burn are u_2 to u_{k+1} and v_0 as well as the right part of the graph. This yields a lower bound for the welfare of an optimal solution.

Furthermore, we have that $s = (v_1', v_2', \ldots, v_k', \emptyset^{|\mathcal{N}|-k})$ is an equilibrium that saves node w and the nodes that are protected by the firefighters. Because at time-step $k+1$ there are no vulnerable nodes left, players i, with $i > k$, have no incentive to deviate from the empty set. Furthermore, we have to argue that for every joint deviation of a coalition of size k, at most $k+1$ nodes can be saved.

We use an inductive argument to show that there is no attractive joint deviation for any coalition of size k into the left part of the graph. Notice that for the left side of the graph, v_1 is connected to all other vulnerable nodes, which means that if player 1 does not protect v_1, then all vulnerable nodes in the left part will be adjacent to the fire in the next time-step. The next player can protect only one extra node, but then everything burns with a total number of two saved nodes. If player 1 protected v_1, node v_2 assumes the role of v_1 for the next time-step, because it is again connected to all other vulnerable nodes. Hence, for time-step i, with $i \leq k$, we have the following situation. If players from 1 to $i-1$ protect nodes v_1 to v_{i-1}, respectively, v_i is connected to all other vulnerable nodes. If player i does not protect v_i, the next player can protect at most one extra node and everything else burns, yielding a number of at most $i+1$ total saved nodes. This implies that in order to save at least $k+1$ nodes in the left part of the graph, players from 1 to k have to protect nodes from v_1 to v_k, respectively, and player $k+1$ has to play an action different from the empty set. Hence, more than k players would have to jointly deviate from strategy profile s.

For the right part of the graph, we can make a symmetric argument, where v_i' assumes the role of v_i for every i, $1 \leq i \leq k$. This yields that the only way to save at least $k+1$ nodes is to play strategy profile s.

Finally, we have to argue that there is no attractive joint deviation considering the case that some nodes from both sides of the graph are protected. Note that both sides of the graph burn in two time-steps if there are no firefighters. This implies that the only way to place firefighters in both parts of the graph is to put one in the left side at the first step and one in the right side at the second, or vice versa. Then, at the third time-step, all other nodes in the part where the firefighter was placed in the second time-step are burning. In the part where the first firefighter was placed, all unprotected nodes are also burning or are adjacent to the fire. This means that only one extra node can be saved in the third time-step and then the process finishes saving at most three nodes.

Now, we can conclude that $s \in E_k$ and we have that $W(s) = k + 1$. This yields a lower bound

$$\text{PoA}_k(\mathcal{G}) \geq \frac{n - 3k - 1}{k + 1} \geq \frac{n}{k + 1} - 3.$$

From this lower bound jointly with the general upper bound shown in Proposition 5.4, it follows that $\text{PoA}_k(\mathcal{G}) = \Theta(\frac{n}{k})$. $\qquad\square$

It is interesting to see that in the same family of graphs $\{G_{PoA}(n, k)\}_{n \geq 2k}$, if $n = \Theta(k)$, then $\text{PoA}_k(\mathcal{G}) = \Theta(1)$. But in general, because k is a constant, we have that $\text{PoA}_k = \Theta(n)$. It seems natural to analyze whether coalitions can improve the quality of equilibria for

certain families of graphs. In the next subsection we address this question with regard with regard to the particular case of families of graphs having constant cut-width.

5.2. Graphs with Constant Cut-Width

In this section we explore the impact of the cut-width of a graph on the PoA for certain coalition sizes. In order to define the cut-width of a graph, we need to introduce some basic concepts.

Definition 5.6. (Cut-width under a linear order.) Given a graph $G = (V, E)$, let $L = (v_0, \ldots, v_{n-1})$ be a linear order of V. The *cut-width of G under the linear order L is* denoted by $\mathrm{cw}(G, L)$ and it is defined as follows:

$$\mathrm{cw}(G, L) = k$$

if, and only if, for every i such that $0 \leq i < n - 1$, the number of edges with one endpoint in $\{v_0, \ldots, v_i\}$ and the other in $\{v_{i+1}, \ldots, v_{n-1}\}$ is at most k and there exists $j, 0 \leq j < n - 1$, such that the number of edges with one endpoint in $\{v_0, \ldots, v_j\}$ and the other in $\{v_{j+1}, \ldots, v_{n-1}\}$ is equal to k.

Definition 5.7. (Cut-width of a graph.) The *cut-width of a graph G is* denoted by $\mathrm{cw}(G)$ and it is defined as follows:

$$\mathrm{cw}(G) = \min\{k \mid \exists L \text{ linear order of } V : \mathrm{cw}(L, G) = k\}.$$

Lemma 5.8. *Let (G, v_0) be an instance of the firefighter problem. Then, there exists a strategy profile such that at most $b(\mathrm{cw}(G))$ nodes are burned for some function b.*

The proof of a more general version of this claim in contained in the proof of Theorem 2 of [10] and brings us into the position of showing the following proposition.

Proposition 5.9. *For any $k \geq 1$, for any graph $G = (V, E)$ such that $\mathrm{cw}(G) = k$, and any $v_0 \in V$, there exists a coalition size b_k such that*

$$\mathrm{PoA}_{b_k}(\mathcal{G}(G, v_0)) \leq 1 + \frac{b_k - 1}{n - b_k}.$$

Proof. Let $\mathcal{G} = \mathcal{G}(G, v_0)$ where $G = (V, E)$, $v_0 \in V$ and $\mathrm{cw}(G) = k$. By Lemma 5.8 there is strategy profile s, such that at most $b_k = b(\mathrm{cw}(G))$ nodes burn. Now we make use of the fact that the number of time-steps before the spreading of the fire stops is fewer or equal to the total number of burned vertices. This is because in each time-step, at least one node has to burn, otherwise the spreading of the fire would be stopped. Hence, we get that with strategy profile s, the fire is contained in at most b_k time-steps. Note that we can place at most one firefighter per time-step, therefore, a coalition of size b_k can apply this strategy profile. Furthermore, only a constant number of nodes burn. Hence, asymptotically, we have that $\mathrm{PoA}_{b_k}(\mathcal{G}) \leq \frac{n-1}{n-b_k}$. $\qquad\qquad \square$

However, we cannot achieve this PoA without coalitions as the following instance shows. Figure 5 shows a family of graphs $G_{cw}(n)$. Note that $n = 4m$. Each t_i is connected

to u_i and v_i; furthermore, we have edges (u_i, v_i) and (v_i, w_i) for each i. Additionally, for each $i < m$, we have edges (t_i, t_{i+1}), (u_i, u_{i+1}), and (v_i, v_{i+1}). A linear layout is given by the horizontal position of the nodes in the figure,

$$(t_1, u_1, v_1, t_2, w_1, u_2, v_2, t_3, w_2, \ldots, t_m, w_{m-1}, u_m, v_m, w_m).$$

It shows that the cut-width of the graph is at most 6, because every vertical line through the graph crosses at most 6 edges. Let $v_0 = u_1$ be the initial fire. Without coalitions, saving the nodes v_1 to v_m is an equilibrium, because each player saves one extra node and cannot do better by switching to another node. Note that only a constant fraction of the nodes is saved, whereas, in the case of coalition, all nodes except a constant number can be saved. Thus, for this class of graphs, constant-size coalitions can improve the PoA.

6. COMPUTATIONAL COMPLEXITY

Let us explore the computational complexity of computing equilibria. More specifically, let us bound the computation time of a best response and the converging time of best response dynamics.

Lemma 6.1. *For any coalition size $k \geq 1$, the computational complexity of finding an equilibrium with respect to a coalition size k of any given non profit firefighter game is polynomial time computable.*

Proof. Let $\mathcal{G} = \mathcal{G}(G, v_0)$ any non profit firefighter game and let $n = |V(G)|$. Note that for a given strategy vector s, we can simulate the spreading of fire in polynomial time and, hence, compute the social welfare $W(s)$ and check whether there are no invalid actions in s. Let $\mathcal{O}(q(n))$, for some polynomial q, denote the computation time.

For a coalition size k, we can find a best response (if it exists) in time $\mathcal{O}(q(n) \cdot n^k)$ by simply trying out every possible joint deviation. Furthermore, we have that the social welfare is strictly increased with every best response, because it equals the utility function of the players. Recall that the social welfare counts the nodes that are saved, so its value cannot exceed $n - 1$. Hence, the best response dynamics converge after at most n iterations.

It follows that we can find an equilibrium in time $\mathcal{O}(p(n) \cdot n^k)$ for some polynomial p. $\qquad\square$

This implies that we can find equilibria in polynomial time in the case of constant size coalitions. This yields to a polynomial time approximation algorithm for the firefighting problem. The computation of a NE with respect a coalition size k can be seen as an approximation algorithm with an approximation ratio that depends on the quality of the equilibrium. Hence, in the case of trees we have shown that we can compute in polynomial time a NE and its quality with respect to the social optimum that is bounded by PoA ≤ 2
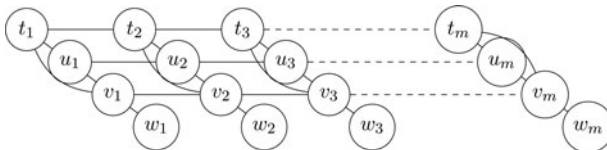

Figure 5 Family of graphs $G_{cw}(n)$ with constant cut-width.

(coalition size 1). In the case of graphs with constant cut-width k, we can compute a NE with $\text{PoA}_{b_k} \leq 1 + \frac{b_k - 1}{n - b_k}$ (coalition size b_k).

7. CONCLUSIONS

We have defined a new strategic game that models the Firefighter Problem. We have shown that for a particular family of graphs $\text{PoA}(\mathcal{G}) = \Theta(n)$. For trees, however, we get a $\text{PoA}(\mathcal{G}) \leq 2$. Furthermore, we have shown that the coalition size has a direct effect on the quality of the equilibria. We have proved that $\text{PoA}_k(\mathcal{G}) = \Theta(\frac{n}{k})$ for a specific family of graph, with k being the coalition size. We have shown that there are topologies where $\text{PoA}_{b_k}(\mathcal{G}) \leq 1 + \frac{b_k - 1}{n - b_k}$ for coalitions of constant size b_k, e.g., graphs with constant cut-width k.

Notice that it is possible to compute an equilibrium in polynomial time, even for constant-size coalitions. This can be done following a best response dynamics. This yields to a polynomial time approximation algorithm for the Firefighter Problem and its approximation ratio equals the PoA of the corresponding game.

We think that the most promising area to explore is the PoA for other restricted classes of graphs, and the impact of the coalition size in those cases. Particularly interesting are classes of graphs within which we can achieve a good PoA with constant-sized coalitions. The results for trees and graphs with constant cut-width are a nice starting step in this direction. In the case of trees, we showed a PoA of 2; the next step would be to investigate how far we can improve the PoA if we allow coalitions. Note that it is not possible to get a PoA of one for trees with constant-sized coalitions unless P = NP, because the firefighter problem is known to be NP-hard, even on trees [17]. However, we can try to find restricted classes of trees within which we can get a PoA of one. Finding the most general description of trees with this property is an open problem [18]. Other classes of graphs we want to consider are grids and graphs with a bounded number of cycles. A related open question is whether the Firefighter Problem is solvable in polynomial time on graphs with path-width two [10].

It also makes sense to analyze the quality of the equilibria with respect to different game-theoretical solution concepts and different social welfare functions. In general, there are other optimization objectives for the Firefighter Problem, for example, minimize the number of time-steps until the fire stops spreading [18]. They have to be reflected in the social welfare function if we want to analyze the quality of the outcomes of the game for those cases.

A natural extension of our model is to allow more than one firefighter per time-step and more than one initial fire. This is a generalization that has also been made for the classical Firefighter Problem, although most results are done for the basic case of one initial fire and one Firefighter per time-step. Many complexity questions in the general case are still open problems, even for trees.

FUNDING

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