RANDOM INTERSECTION GRAPH PROCESS

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Abstract Vertices of an affiliation network are linked to features and two vertices are declared adjacent whenever they share a common feature. We introduce a random intersection graph process aimed at modeling sparse evolving affiliation networks. We establish the asymptotic degree distribution and provide explicit asymptotic formulas for assortativity and clustering coefficients and show how these edge dependence characteristics vary over time.

1. INTRODUCTION

Affiliation networks are an important class of social networks: members of a network tend to establish relations if they share some common interests or features [12, 17]. For example, for marketing purposes, customers of an internet shop (users of a video-sharing website) can be considered related to each other if they have purchased the same or similar items (e.g., downloaded the same movie). In this article we consider the fitness model of an affiliation network. Vertices (members of the network) and features are assigned independent fitness factors, and a vertex is linked to a feature with probability proportional to the product of their fitness factors. This probability also depends on the arrival times of the vertex and the feature and is set at zero whenever the difference between these arrival times exceeds some threshold value. We show that the network model admits a *power-law degree distribution* and nontrivial *clustering* and *assortativity* coefficients.

Given nonnegative weights $x = \{x_i\}_{i \geq 1}$ and $y = \{y_j\}_{j \geq 1}$, and a nondecreasing positive sequence $\{\tau(t)\}_{t \geq 1}$ satisfying $\lim_{t \rightarrow +\infty} \tau(t) = +\infty$, let $H_{x,y}$ be the random bipartite graph with bipartition $V = \{v_1, v_2, \dots\}$ and $W = \{w_1, w_2, \dots\}$, where edges $\{w_i, v_j\}$ are inserted independently and with probabilities

$$p_{ij} = \min \left\{ 1, \frac{x_i y_j}{\sqrt{i j}} \right\} \mathbb{I}_{\{a\tau(j) \leq i \leq b\tau(j)\}}. \quad (1.1)$$

Here, $b > a > 0$ are fixed numbers. $H_{x,y}$ defines the graph $G_{x,y}$ on the vertex set V such that any $u, v \in V$ are declared adjacent (denoted $u \sim v$) whenever they have a common neighbor in $H_{x,y}$.

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Consider, for example, a library where items $w_1, w_2, \dots$ are acquired one after another, and where users $v_1, v_2, \dots, v_j, \dots$ are registered at times $j = 1, 2, \dots$. User v_j picks at random items from the “contemporary literature collection” $[a\tau(j), b\tau(j)] = \{w_i : a\tau(j) \leq i \leq b\tau(j)\}$ relevant to its arrival time j . An item w_i is picked with probability proportional to the activity y_j of the user v_j and the attractiveness x_i of the item w_i , cf. (1.1). The realized bipartite graph $H_{x,y}$ represents “library” records, whereas realized $G_{x,y}$ represents adjacency relations between users in the resulting affiliation network.

Theoretical analysis of such a network becomes simpler if we impose some regularity conditions on the weight sequences x and y . A convenient assumption is that x and y are realized values of i.i.d sequences $X = \{X_i\}_{i \geq 1}$ and $Y = \{Y_j\}_{j \geq 1}$. In this way we obtain random graphs $H_{X,Y}$ and $G_{X,Y}$. The parameters of the model are the probability distributions of X_1, Y_1 , function τ (the speed of item acquisition), and cutoffs $a < b$, which, together with $\tau(j)$, determine the interval $[a\tau(j), b\tau(j)]$, which we also interpret as the lifespan of a user v_j . In what follows, we mainly consider the case where $\tau(t) = t^\nu$, $\nu > 0$. In this case our choice of weights $1/\sqrt{ij}$ (together with moment conditions on X and Y) ensures that the resulting random graph $G_{X,Y}$ is sparse, i.e., the degree sequence $\{d(v_j)\}_{j \geq 1}$ is stochastically bounded.

Random graph $G_{X,Y}$ is aimed at modeling sparse affiliation networks that evolve in time, such as, e.g., the actor network, where two actors are declared adjacent whenever they have played in the same movie, a collaboration network where two authors are declared adjacent if they have coauthored an article. Such networks display a natural bipartite structure: actors are linked to films, authors are linked to articles. An underlying bipartite structure seems to be present in many social networks: members of a network become acquainted because they share some common interests. The bipartite structure is helpful in explaining the clustering property (network transitivity), and the observed fact that degrees of adjacent vertices are positively correlated (network assortativity), [17, 20].

We note that each vertex of the graph $G_{X,Y}$ can be identified with the random subset of W , consisting of items selected by that vertex, and two vertices are adjacent in $G_{X,Y}$ whenever their subsets intersect. Graphs describing such adjacency relations between members of a *finite* family $V_n = \{v_1, \dots, v_n\}$ of random subsets of a given *finite* set $W_m = \{w_1, \dots, w_m\}$ have been introduced in [14], see also [10]. They are called *random intersection graphs*. We remark that random intersection graphs reproduce empirically observed clustering properties of the actor network with remarkable accuracy ([2], [6]). Unfortunately, they do not account for the evolving nature of the network (actors who acted in 2013 are unlikely to be adjacent to those who acted in 1913) and, therefore, cannot explain how various characteristics of an evolving network vary over time. This drawback of the static model has motivated our interest in evolving random intersection graph $G_{X,Y}$. A related empirical study showing how various characteristics of an evolving collaboration network changed over a 100-year period is presented in [15], see also [16].

The random graph $G_{X,Y}$ can be considered as a random process evolving in time, where the vertex v_t that arrived at time t can establish adjacency relations only with *contemporaries* v_s such that the intervals $[a\tau(t), b\tau(t)]$ and $[a\tau(s), b\tau(s)]$ (lifespans of v_t and v_s) intersect. We show that the random intersection graph process admits an asymptotic power-law distribution of the degree $d(v_t)$ of a vertex v_t as $t \rightarrow +\infty$. Moreover, we give an explicit description of the asymptotic degree distribution. Furthermore, we show that $G_{X,Y}$ admits a nontrivial *clustering* and *assortativity* coefficients and calculate their first-order asymptotics.

The intuition behind these results is explained as follows. We first observe that in choosing inhomogeneous weight sequences x and y , one may expect to obtain an inhomogeneous degree sequence of the graph $G_{x,y}$: vertices with larger weights attract larger numbers of neighbors. Consequently, in the case where the probability distributions of X_1 and Y_1 have heavy tails, we obtain a heavy-tailed asymptotic degree distribution. Second, we observe that if the set $W(t)$ of items selected by a user v_t is (stochastically) bounded and the lifespans of two neighbors of v_t , say v_s and v_u , intersect, then, with a nonvanishing probability, v_s and v_u share an item from $W(t)$. Consequently, the conditional probability $\alpha_{t|su} = \mathbf{P}(v_s \sim v_u | v_s \sim v_t, v_t \sim v_u)$, called the clustering coefficient, is positive and bounded away from zero. In particular, the underlying bipartite graph structure serves as a clustering mechanism. A similar argument applies to the assortativity coefficient (Pearson's correlation coefficient between degrees of adjacent vertices)

$$r_{s,t} = \frac{\mathbf{E}_{st}d(v_s)d(v_t) - \mathbf{E}_{st}d(v_s)\mathbf{E}_{st}d(v_t)}{\sqrt{\mathbf{Var}_{st}d(v_s)\mathbf{Var}_{st}d(v_t)}}. \quad (1.2)$$

Here, an item shared by adjacent vertices $v_s \sim v_t$ attracts a number of common neighbors of v_s and v_t . This makes the correlation coefficient positive and bounded away from zero. Here, $\mathbf{E}_{st}$ denotes the conditional expectation given the event $v_s \sim v_t$ and $\mathbf{Var}_{st}d(v_s) = \mathbf{E}_{st}d^2(v_s) - (\mathbf{E}_{st}d(v_s))^2$.

2. RESULTS

2.1. Degree

For $\tau(t)$ growing linearly in t we obtain a compound probability distribution in the limit. We remark that in this case $G_{X,Y}$ admits nontrivial asymptotic clustering coefficients, see (2.6), (2.7), (2.8) below. For $\tau(t)$ growing faster than linearly in t , we obtain a mixed Poisson asymptotic degree distribution. In this case clustering coefficients vanish.

Theorem 2.1. *Let $b > a > 0$. Let $\tau(t) = t$. Suppose that $\mathbf{E}X_1^2 < \infty$ and $\mathbf{E}Y_1 < \infty$. Set $\lambda_1 = \lambda_1(Y_1) = 2(b^{1/2} - a^{1/2})(\mathbf{E}X_1)Y_1$ and $\lambda_2 = \lambda_2(X_1) = 2(a^{-1/2} - b^{-1/2})(\mathbf{E}Y_1)X_1$.*

For $t \rightarrow +\infty$, the random variable $d(v_t)$ converges in distribution to the random variable

$$d_* = \sum_{j=1}^{\Lambda_1} \varkappa_j, \quad (2.1)$$

where $\varkappa_1, \varkappa_2, \dots$ are independently and identically distributed random variables independent of the random variable Λ_1 . They are distributed as follows. For $r = 0, 1, 2, \dots$, we have

$$\mathbf{P}(\varkappa_1 = r) = \frac{r+1}{\mathbf{E}\Lambda_2} \mathbf{P}(\Lambda_2 = r+1), \quad \text{where} \quad \mathbf{P}(\Lambda_i = r) = \mathbf{E} e^{-\lambda_i} \frac{\lambda_i^r}{r!}, \quad i = 1, 2. \quad (2.2)$$

We remark that the second moment condition $\mathbf{E}X_1^2 < \infty$ of Theorem 2.1 seems to be redundant and could perhaps be waived.

Theorem 2.2. Let $b > a > 0$ and $\nu > 1$. Let $\tau(t) = t^\nu$. Suppose that $\mathbf{E}X_1^2 < \infty$ and $\mathbf{E}Y_1 < \infty$. Set $\lambda_3 = \lambda_3(Y_1) = \gamma(\mathbf{E}X_1^2)(\mathbf{E}Y_1)Y_1$, where $\gamma = 4\nu(b^{1/2\nu} - a^{1/2\nu})(a^{-1/2\nu} - b^{-1/2\nu})$.

For $t \rightarrow +\infty$, the random variable $d(v_t)$ converges in distribution to the random variable Λ_3 having the probability distribution

$$\mathbf{P}(\Lambda_3 = r) = \mathbf{E} e^{-\lambda_3} \frac{\lambda_3^r}{r!}, \quad r = 0, 1, 2, \dots \quad (2.3)$$

Remark 2.3. The probability distributions Λ_i , $i = 1, 2, 3$, are Poisson mixtures. One way to sample from the distribution of Λ_i is to generate random variable λ_i and then, given λ_i , to generate a Poisson random variable with the parameter λ_i . In Theorem 2.1 we obtain a power-law asymptotic degree distribution provided that the heavier of the tails $t \rightarrow \mathbf{P}(\varkappa_1 > t)$ and $t \rightarrow \mathbf{P}(\Lambda_1 > t)$ has a power law, see [9]. In Theorem 2.2 we obtain a power-law asymptotic degree distribution provided that Y_1 has a power law. Indeed, since for large λ the Poisson distribution $\mathcal{P}(\lambda)$ is tightly concentrated around its mean λ , we have that, given a large value of Y_1 , the conditional distribution of Λ_3 is concentrated around $\lambda_3 = \gamma(\mathbf{E}X_1^2)(\mathbf{E}Y_1)Y_1$. Now, since the latter random variable has a power law, we conclude that Λ_3 has a power law with the same exponent as Y_1 .

Remark 2.4. The result of Theorem 2.2 extends to a more general class of increasing nonnegative functions τ . In particular, assuming that

$$\lim_{t \rightarrow +\infty} \frac{t}{\tau(t)} = 0, \quad \sup_{t > 1} \frac{\tau^{-1}(2t)}{\tau^{-1}(t)} < \infty, \quad (2.4)$$

and that there exists a finite limit

$$\gamma^* = \lim_{t \rightarrow +\infty} t^{-1/2} \sum_{a\tau(t) \leq i \leq b\tau(t)} i^{-1} \sum_{j: a\tau(j) \leq i \leq b\tau(j)} j^{-1/2},$$

we obtain the convergence in distribution of $d(v_t)$ to Λ_3 defined by (2.3) with $\lambda_3 = \gamma^*(\mathbf{E}X_1^2)(\mathbf{E}Y_1)Y_1$. Here, τ^{-1} denotes the inverse of τ (i.e., $\tau(\tau^{-1}(t)) = t$).

Remark 2.5. The function $\tau(t) = t \ln t$, which grows slower than any power t^ν , $\nu > 1$, satisfies conditions of Remark 2.4 with $\gamma^* = 4(a^{-1/2} - b^{-1/2})(b^{1/2} - a^{1/2})$. Furthermore, the functions $\tau_1(t) = e^{\ln^2 t}$ and $\tau_2(t) = e^t$, which grow faster than any power t^ν , satisfy conditions of Remark 2.4 with $\gamma^* = 0$ and now the asymptotic degree distribution is degenerate. The function $\tau(t) = t^\nu$, for $0 < \nu < 1$, does not satisfy conditions of Remark 2.4. In this case, the asymptotic degree distribution is again degenerate, i.e., for $\mathbf{E}X_1 < \infty$ and $\mathbf{E}Y_1 < \infty$ we have $d(v_t) = o_P(1)$ as $t \rightarrow +\infty$. This fact is obtained in the same way as the statement (i) of Theorem 1 of [3].

2.2. Clustering

We assume that $s < t < u$ and consider the conditional probabilities $\alpha_{s|tu}$, $\alpha_{t|su}$ and $\alpha_{u|st}$. These probabilities are all different and, given $0 < a < b$, they depend mainly on the ratios s/t , s/u , and t/u . Denote $p_\Delta = p_\Delta(s, t, u) = \mathbf{P}(v_s \sim v_t, v_s \sim v_u, v_t \sim v_u)$ the probability that v_s, v_t, v_u make up a triangle.

Theorem 2.6. Let $b > a > 0$. Let $\tau(t) = t$. Suppose that $\mathbf{E}X_1^3 < \infty$ and $\mathbf{E}Y_1^2 < \infty$. Assume that $s, t, u \rightarrow +\infty$ so that $s < t < u$ and $\lceil au \rceil \leq \lfloor bs \rfloor$. We have

$$p_\Delta = \frac{a_3 b_1^3}{\sqrt{stu}} \left(\frac{2}{\sqrt{au}} - \frac{2}{\sqrt{bs}} \right) + o(t^{-2}), \quad (2.5)$$

$$\alpha_{t|su} = \frac{p_\Delta}{p_\Delta + a_2^2 b_1^2 b_2 t^{-1} (su)^{-1/2} \delta_{t|su}} + o(1), \quad (2.6)$$

$$\alpha_{s|tu} = \frac{p_\Delta}{p_\Delta + a_2^2 b_1^2 b_2 s^{-1} (tu)^{-1/2} \delta_{s|tu}} + o(1), \quad (2.7)$$

$$\alpha_{u|st} = \frac{p_\Delta}{p_\Delta + a_2^2 b_1^2 b_2 u^{-1} (st)^{-1/2} \delta_{u|st}} + o(1). \quad (2.8)$$

Here, we denote $a_i = \mathbf{E}X_1^i$, $i = 2, 3$, and $b_j = \mathbf{E}Y_1^j$, $j = 1, 2$, and

$$\begin{aligned} \delta_{t|su} &= \ln(u/t) \ln(t/s) + \ln(u/t) \ln(bs/au) + \ln(t/s) \ln(bs/au) + \ln^2(bs/au), \\ \delta_{s|tu} &= \ln(u/t) \ln(bs/au) + \ln^2(bs/au), \\ \delta_{u|st} &= \ln(t/s) \ln(bs/au) + \ln^2(bs/au). \end{aligned}$$

We remark that the condition $\lceil au \rceil \leq \lfloor bs \rfloor$ of Theorem 2.6 excludes the trivial case where $p_\Delta \equiv 0$. Indeed, for $s < u$, the converse inequality $\lceil au \rceil > \lfloor bs \rfloor$ means that the lifetimes of v_s and v_u do not intersect and, therefore, we have $\mathbf{P}(v_s \sim v_u) \equiv 0$. In addition, the inequality $\lceil au \rceil \leq \lfloor bs \rfloor$ implies that positive numbers $\delta_{t|su}$, $\delta_{s|tu}$, $\delta_{u|st}$ are bounded from above by a constant (depending only on a and b).

2.3. Assortativity

We assume that $\tau(t) = t$ and consider the sequence of random variables $\{d(v_t)\}_{t \geq 1}$. From Theorem 2.1 we know about the possible limiting distributions for $d(v_t)$. Moreover, from the fact that $G_{X,Y}$ is sparse, we can conclude that, for any given k , the random variables $d(v_t), d(v_{t+1}), \dots, d(v_{t+k})$ are asymptotically independent as $t \rightarrow +\infty$. An interesting question is about the statistical dependence between $d(v_s)$ and $d(v_t)$ if we know, in addition, that vertices v_s and v_t are adjacent in $G_{X,Y}$. We assume that $s < t$ and let $s, t \rightarrow +\infty$ so that $bs - at \rightarrow +\infty$. Note that the latter condition ensures that the shared lifetime of v_s and v_t tends to infinity as $s, t \rightarrow +\infty$. In this case we obtain that conditional moments

$$\mathbf{E}_{st} d(v_s) = \mathbf{E}_{st} d(v_t) + o(1) = \delta_1 + o(1), \quad (2.9)$$

$$\mathbf{E}_{st} d^2(v_s) = \mathbf{E}_{st} d^2(v_t) + o(1) = \delta_2 + o(1),$$

$$\mathbf{E}_{st} d(v_s) d(v_t) = \delta_2 - \Delta + o(1)$$

are asymptotically constant. Here, $\Delta = h_1^{-1}(2h_3 + 2h_5 + 4(h_6 - h_7))$ and

$$\delta_1 = 1 + h_1^{-1}(h_2 + 2h_3), \quad \delta_2 = 1 + h_1^{-1}(3h_2 + 6h_3 + h_4 + 6h_5 + 4h_6).$$

Furthermore, the quantities h_i are polynomials in moments $a_k = \mathbf{E}X_1^k$ and $b_k = \mathbf{E}Y_1^k$

$$\begin{aligned} h_1 &= a_2 b_1^2, & h_2 &= a_3 b_1^3 \tilde{\gamma}, & h_3 &= a_2^2 b_1^2 b_2 \tilde{\gamma}(\sqrt{b} - \sqrt{a}), \\ h_4 &= a_4 b_1^4 \tilde{\gamma}^2, & h_5 &= a_2 a_3 b_1^3 b_2 \tilde{\gamma}^2(\sqrt{b} - \sqrt{a}), \\ h_6 &= a_2^3 b_1^3 b_3 \tilde{\gamma}^2(\sqrt{b} - \sqrt{a})^2, & h_7 &= a_2^3 b_1^2 b_2^2 \tilde{\gamma}^2(\sqrt{b} - \sqrt{a})^2, \end{aligned} \quad (2.10)$$

and $\tilde{\gamma} = 2(a^{-1/2} - b^{-1/2})$.

From (1.2) and (2.9) we obtain that the assortativity coefficient

$$r_{st} = 1 - \frac{\Delta}{\delta_2 - \delta_1^2} + o(1) \quad (2.11)$$

is asymptotically constant.

2.4. Related Work

The degree distribution of the typical vertex of *finite* random intersection graphs has been studied by several authors, see, e.g., [2], [8], [13], [18], the clustering properties have been studied in [2], [5] [8], [11]. Assortativity coefficient has been evaluated in [5].

In a recent article [4], a related intersection graph process is considered, where a newly arrived user is linked preferentially to already highly popular items, thus, further increasing their popularity/attractiveness measured by the number of users that have chosen this item. We note that the spatial preferred attachment model [1] can also be viewed as a geometric intersection graph process.

3. PROOFS

We first introduce some notation. Then we prove Theorems 2.1 and 2.2. The proof of Remark 2.4 goes along the lines of the proof of Theorem 2.2 and is omitted. Theorem 2.6 and relations (2.9) are shown in the accompanying study [7] (see pp. 100–104 of [7]). In order to avoid repetition we do not present their proofs here.

In the proofs of Theorems 2.1 and 2.2, we apply the approach used in [3]. Here, the main technical difference between the static model $G(P_1, P_2, n, m)$ of [3] and the process $G_{X,Y}$ is that the scaling $\frac{1}{\sqrt{ij}}$ of the edge inclusion probability p_{ij} of the underlying bipartite graph of $G_{X,Y}$ depends on the arrival times i and j of item w_i and user v_j , whereas the scaling $\frac{1}{\sqrt{nm}}$ of the corresponding edge inclusion probability p'_{ij} of the static model is independent of i and j . The dependence on arrival times i and j adds some more complexity to the proofs and affects the parameters of random variables $\lambda_1, \lambda_2, \lambda_3$ of (2.2), (2.3).

We denote $a_k = \mathbf{E}X_1^k$ and $b_k = \mathbf{E}Y_1^k$. Throughout the proof, limits are taken as $t \rightarrow +\infty$, if not stated otherwise. By c we denote positive numbers that may depend only on a, b and τ . We remark that c may attain different values in different places. We say that a sequence of random variables $\{\zeta_t\}_{t \geq 1}$ converges to zero in probability (denoted $\zeta_t = o_P(1)$) whenever $\limsup_t \mathbf{P}(|\zeta_t| > \varepsilon) = 0$ for each $\varepsilon > 0$. The sequence $\{\zeta_t\}_{t \geq 1}$ is called stochastically bounded (denoted $\zeta_t = O_P(1)$) whenever for each $\delta > 0$ there exists $N_\delta > 0$ such that $\limsup_t \mathbf{P}(|\zeta_t| > N_\delta) < \delta$.

Time intervals

$$T_t = \{i : a\tau(t) \leq i \leq b\tau(t)\}, \quad T_t^* = \{j : a\tau(j) \leq i \leq b\tau(j)\} \quad (3.1)$$

can be interpreted as lifetimes of the actor v_t and attribute w_i respectively. Here and below elements of V are called actors, elements of W are called attributes. The oldest and youngest actors that may establish a communication link with v_t are denoted v_{t-} and v_{t+} . Here

$$t_- = \min\{j : T_j \cap T_t \neq \emptyset\}, \quad t_+ = \max\{j : T_j \cap T_t \neq \emptyset\}.$$

The event “edge $\{w_i, v_j\}$ is present in $H_{X,Y}$ ” is denoted $w_i \rightarrow v_j$. Introduce random variables

$$\begin{aligned} \mathbb{I}_{ij} &= \mathbb{I}_{\{w_i \rightarrow v_j\}}, & \mathbb{I}_i &= \mathbb{I}_{it}, & U_i &= \sum_{j \in T_i^* \setminus \{t\}} \mathbb{I}_{ij}, & L &= L_t = \sum_{i \in T_t} U_i \mathbb{I}_i, \\ B_k(I) &= \sum_{j \in I} Y_j^k j^{-k/2}, & A_k(I) &= \sum_{i \in I} X_i^k i^{-k/2}, & I &\subset \mathbb{N}, \end{aligned} \quad (3.2)$$

$$\lambda_{ij} = X_i Y_j / \sqrt{ij}, \quad Q_{XY}(t) = \sum_{i \in T_t} \lambda_{it} \sum_{j \in T_i^* \setminus \{t\}} \lambda_{ij} \min\{1, \lambda_{ij}\}. \quad (3.3)$$

We remark that U_i counts all neighbors of w_i in $H_{X,Y}$ belonging to the set $V \setminus \{v_t\}$, and L_t counts all paths of length 2 in $H_{X,Y}$ starting from v_t . Introduce events

$$\mathcal{A}_t = \{\lambda_{it} \leq 1, i \in T_t\}, \quad \mathcal{B}_t(\varepsilon) = \{Y_j \leq \varepsilon^2 j, j \in [t_-, t_+] \setminus \{t\}\}, \quad \varepsilon > 0.$$

By $\tilde{\mathbf{P}}$ and $\tilde{\mathbf{E}}$ we denote the conditional probability and expectation given X, Y . The conditional probability and expectation given Y is denoted $\mathbf{P}_X$ and $\mathbf{E}_X$. By $\mathbf{P}_t$ and $\mathbf{E}_t$ we denote the conditional probability and expectation given Y_t . By $d_{TV}(\zeta, \xi)$ we denote the total variation distance between the probability distributions of random variables ζ and ξ . In the case where ζ, ξ and X, Y are defined on the same probability space, we denote by $\tilde{d}_{TV}(\zeta, \xi)$ the total variation distance between the conditional distributions of ζ and ξ given X, Y .

In the proof, we use the following simple fact. For a uniformly bounded sequence of nonnegative random variables $\{\zeta_t\}_{t \geq 1}$ (i.e., such that for some nonrandom $h > 0$ and each t the inequalities $0 \leq \zeta_t < h$ hold almost surely) we have

$$\zeta_t = o_P(1) \quad \Rightarrow \quad \mathbf{E}\zeta_t = o(1). \quad (3.4)$$

In particular, given a sequence of bivariate random vectors $\{(\phi_t, \psi_t)\}_{t \geq 1}$, defined on the same probability space as X, Y , we have

$$\tilde{d}_{TV}(\phi_t, \psi_t) = o_P(1) \quad \Rightarrow \quad d_{TV}(\phi_t, \psi_t) = o(1). \quad (3.5)$$

Finally, we note that the notation introduced in the proof of a particular lemma or theorem is valid only for that proof.

3.1. Proof of Theorem 2.1

Before the proof we collect auxiliary results. For $\tau(t) := t$ and T_t, T_t^* defined in (3.1), we have

$$\begin{aligned} \sum_{i \in T_t} i^{-1/2} &= t^{1/2} \gamma_1 + r t^{-1/2}, & \sum_{j \in T_t^*} j^{-1/2} &= i^{1/2} \gamma_2 + r' i^{-1/2}, \\ \gamma_1 &:= 2(b^{1/2} - a^{1/2}), & \gamma_2 &:= 2(a^{-1/2} - b^{-1/2}), \end{aligned} \quad (3.6)$$

where $|r|, |r'| \leq c$.

Lemma 3.1. *Let $t \rightarrow +\infty$. Assume that $\mathbf{E}X_1^2 < \infty$ and $\mathbf{E}Y_1 < \infty$. We have*

$$\forall \varepsilon > 0 \quad \mathbf{P}(\mathcal{B}_t(\varepsilon)) = 1 - o(1), \quad (3.7)$$

$$t^{-1}B_2([t_-, t_+] \setminus \{t\}) = o_P(1), \quad (3.8)$$

$$\mathbf{P}(d(v_t) \neq L_t) = o(1), \quad (3.9)$$

$$\mathbf{P}(\mathcal{A}_t) = 1 - o(1), \quad (3.10)$$

$$Q_{XY}(t) = o_P(1), \quad \mathbf{E}Q_{XY}(t) = o(1). \quad (3.11)$$

For any integers $t > a^{-1}(b + b^{-1})$ and $i \in T_t$, and any $0 < \varepsilon < 1$ we have

$$|\mathbf{E}A_1(T_t) - a_1\gamma_1 t^{1/2}| \leq ca_1 t^{-1/2}, \quad |\mathbf{E}B_1(T_t^* \setminus \{t\}) - b_1\gamma_2 i^{1/2}| \leq cb_1 i^{-1/2}, \quad (3.12)$$

$$\mathbf{E}|B_1(T_t^* \setminus \{t\}) - b_1\gamma_2 i^{1/2}| \mathbb{I}_{\mathcal{B}_t(\varepsilon)} \leq ci^{1/2}(\varepsilon b_1^{1/2} + \mathbf{E}Y_1 \mathbb{I}_{\{Y_1 > \varepsilon^2 t_-\}}) + cb_1 i^{-1/2}, \quad (3.13)$$

$$\mathbf{E}|A_1(T_t) - a_1\gamma_1 t^{1/2}| \leq ca_2^{1/2}. \quad (3.14)$$

Proof of Lemma 3.1. Proof of (3.7). We estimate the probability of the complement event $\overline{\mathcal{B}}_t(\varepsilon)$ using the union bound and Markov's inequality

$$\mathbf{P}(\overline{\mathcal{B}}_t(\varepsilon)) \leq \sum_{t_- \leq j \leq t_+} \mathbf{P}(Y_j > \varepsilon^2 j) = t_+ \mathbf{P}(Y_1 > \varepsilon^2 t_-) \leq \varepsilon^{-2}(t_+/t_-) \mathbf{E}Y_1 \mathbb{I}_{\{Y_1 > \varepsilon^2 t_-\}} = o(1).$$

Here we estimate $t_+/t_- \leq c$ and invoke the bound $\mathbf{E}Y_1 \mathbb{I}_{\{Y_1 > s\}} = o(1)$, for $s \rightarrow +\infty$.

Proof of (3.8). Denote $\hat{b}_2(t) = t^{-2} \sum_{1 \leq j \leq t} Y_j^2$. We note that $\mathbf{E}Y_1 < \infty$ implies $\hat{b}_2(t) = o_P(1)$. The latter bound in combination with the simple inequality $t_+/t_- \leq c$ implies (3.8).

Proof of (3.10). Let $\overline{\mathcal{A}}_t$ denote the complement event to $\mathcal{A}_t$. We have, by the union bound and Markov's inequality,

$$\mathbf{P}_t(\overline{\mathcal{A}}_t) \leq \sum_{i \in T_t} \mathbf{P}_t(\lambda_{it} \geq 1) \leq \sum_{i \in T_t} (it)^{-1} Y_t^2 a_2 \leq ca_2 t^{-1} Y_t^2.$$

We obtain the bound $\mathbf{P}_t(\overline{\mathcal{A}}_t) = o(1)$, which implies (3.10), see (3.4).

Proof of (3.9). In view of (3.4) it suffices to show that $\mathbf{P}_X(d(v_t) \neq L_t) = o_P(1)$. We note that $d(v_t) \neq L_t$ if and only if $S \geq 1$, where $S = \sum' \mathbb{I}_{i_1} \mathbb{I}_{i_2} \mathbb{I}_{i_1 j} \mathbb{I}_{i_2 j}$. Here, we denote $\sum' = \sum_{\{i_1, i_2\} \subset T_t} \sum_{j \in T_{i_1}^* \cap T_{i_2}^*, j \neq t}$. Observing that

$$\mathbf{E}_X \mathbb{I}_{i_1} \mathbb{I}_{i_2} \mathbb{I}_{i_1 j} \mathbb{I}_{i_2 j} = \mathbf{E}_X p_{i_1 t} p_{i_2 t} p_{i_1 j} p_{i_2 j} \leq a_2^2 Y_t^2 Y_j^2 / (i_1 i_2 t j),$$

we obtain, by Markov's inequality,

$$\mathbf{P}_X(d(v_t) \neq L_t) = \mathbf{P}_X(S \geq 1) \leq \mathbf{E}_X S \leq a_2^2 Y_t^2 t^{-1} \sum' Y_j^2 (i_1 i_2 j)^{-1}. \quad (3.15)$$

The simple bound $\sum_{\{i_1, i_2\} \subset T_t} \frac{1}{i_1 i_2} \leq c$ implies $\sum' Y_j^2 (i_1 i_2 j)^{-1} \leq c B_2([t_-, t_+] \setminus \{t\})$. Now, by (3.8) the right-hand side of (3.15) tends to zero in probability.

Proof of (3.11). Denote $\hat{X}_i = \max\{X_i, 1\}$, $\hat{Y}_j = \max\{Y_j, 1\}$, and let $\hat{Q}_{XY}(t)$ denote the sum in (3.3), where λ_{ij} is replaced by $\hat{\lambda}_{ij} = \hat{X}_i \hat{Y}_j / \sqrt{ij}$. We observe that $\mathbf{E}X_1^2 < \infty$ and $\mathbf{E}Y_1 < \infty$ imply

$$a_\varphi := \mathbf{E}\hat{X}_1^2\varphi(\hat{X}_1) < \infty, \quad b_\varphi := \mathbf{E}\hat{Y}_1\varphi(\hat{Y}_1) < \infty,$$

for some positive increasing function $\varphi : [1, +\infty) \rightarrow [0, +\infty)$ satisfying $\varphi(u) \rightarrow +\infty$ as $u \rightarrow +\infty$ (clearly, $\varphi(\cdot)$ depends on the distributions of X_1 and Y_1). In addition, we can choose φ satisfying $\varphi(u) \leq u$ and $\varphi(su) \leq \varphi(s)\varphi(u)$, for $s, u \geq 1$. From these inequalities, one derives the inequality $\min\{1, \hat{\lambda}_{ij}\} \leq \varphi(\hat{X}_i)\varphi(\hat{Y}_j)/\varphi(\sqrt{ij})$ (see formula (13) in [3]). The latter inequality implies

$$\hat{Q}_{XY}(t) \leq \hat{Y}_t Q_{XY}^*(t), \quad Q_{XY}^*(t) := \sum_{i \in T_t} \frac{\hat{X}_i^2 \varphi(\hat{X}_i)}{\sqrt{ti}} \sum_{j \in T_i^* \setminus \{t\}} \frac{\hat{Y}_j \varphi(\hat{Y}_j)}{\sqrt{ij}} \frac{1}{\varphi(\sqrt{ij})}.$$

Furthermore, for $i \in T_t$ and $j \in T_i^*$ we have $ij \geq \lfloor at \rfloor t_- =: t_*^2$, and $t_* \rightarrow +\infty$ as $t \rightarrow +\infty$. Hence,

$$\mathbf{E}Q_{XY}^*(t) \leq \frac{1}{\varphi(t_*)} a_\varphi b_\varphi \sum_{i \in T_t} \frac{1}{\sqrt{ti}} \sum_{j \in T_i^*} \frac{1}{\sqrt{ij}} = o\left(\frac{1}{\varphi(t_*)}\right) = o(1).$$

This bound together with the inequalities $Q_{XY}(t) \leq \hat{Q}_{XY}(t) \leq \hat{Y}_t Q_{XY}^*(t)$ shows (3.11).

Proof of (3.12). These inequalities follow from (3.6).

Proof of (3.13). Denote $\Delta = |\tilde{b} - b_1 \gamma_2 i^{1/2}|$, where $\tilde{b}$ denotes the sum $B_1(T_i^* \setminus \{t\})$, but with Y_j replaced by $\tilde{Y}_j = Y_j \mathbb{I}_{\{Y_j \leq \varepsilon^2 j\}}$, $j \in T_i^* \setminus \{t\}$. We have

$$|B_1(T_i^* \setminus \{t\}) - b_1 \gamma_2 i^{1/2}|_{\mathcal{B}_t(\varepsilon)} = \Delta \mathbb{I}_{\mathcal{B}_t(\varepsilon)} \leq \Delta \leq \Delta_1 + \Delta_2 + \Delta_3, \quad (3.16)$$

where we denote $\Delta_1 = |\tilde{b} - \mathbf{E}\tilde{b}|$, $\Delta_2 = |\mathbf{E}\tilde{b} - \mathbf{E}B_1(T_i^* \setminus \{t\})|$, $\Delta_3 = |\mathbf{E}B_1(T_i^* \setminus \{t\}) - b_1 \gamma_2 i^{1/2}|$. Next, we evaluate $\mathbf{E}\Delta_1$ and Δ_2 :

$$(\mathbf{E}\Delta_1)^2 \leq \mathbf{E}(\Delta_1^2) \leq \sum_{j \in T_i^* \setminus \{t\}} j^{-1} \mathbf{E}\tilde{Y}_j^2 \leq \varepsilon^2 b_1 |T_i^*|, \quad (3.17)$$

$$\Delta_2 \leq \sum_{j \in T_i^* \setminus \{t\}} j^{-1/2} \mathbf{E}Y_j \mathbb{I}_{\{Y_j > \varepsilon^2 j\}} \leq \mathbf{E}Y_1 \mathbb{I}_{\{Y_1 > \varepsilon^2 t_-\}} \sum_{j \in T_i^* \setminus \{t\}} j^{-1/2}. \quad (3.18)$$

In (3.17) we first apply Cauchy–Schwartz, then use the linearity of variance of a sum of independent random variables, and finally apply the inequality $\mathbf{Var}\tilde{Y}_j \leq \mathbf{E}\tilde{Y}_j^2 \leq j\varepsilon^2 \mathbf{E}Y_j$. Invoking (3.12), (3.17), (3.18) in (3.16) and using (3.6) and $|T_i^*| \leq ci$, we obtain (3.13).

Proof of (3.14). We have $\mathbf{E}|A_1(T_t) - a_1 \gamma_1 t^{1/2}| \leq \mathbf{E}\tilde{\Delta}_1 + \tilde{\Delta}_2$, where $\tilde{\Delta}_1 := |A_1(T_t) - \mathbf{E}A_1(T_t)|$ and $\tilde{\Delta}_2 = |\mathbf{E}A_1(T_t) - a_1 \gamma_1 t^{1/2}|$, and invoke the inequalities

$$(\mathbf{E}\tilde{\Delta}_1)^2 \leq \mathbf{E}\tilde{\Delta}_1^2 = \sum_{i \in T_t} j^{-1} (a_2 - a_1^2) \leq ca_2$$

and $\tilde{\Delta}_2 \leq ca_1 \leq ca_2^{1/2}$, see (3.12). $\square$

Inequality (3.19) below is referred to as Le Cam's inequality, see e.g., [19].

Lemma 3.2. Let $S = \mathbb{I}_1 + \mathbb{I}_2 + \cdots + \mathbb{I}_n$ be the sum of independent random indicators with probabilities $\mathbf{P}(\mathbb{I}_i = 1) = p_i$. Let Λ be a Poisson random variable with mean $p_1 + \cdots + p_n$. The total variation distance between the distributions P_S of S and P_Λ of Λ

$$d_{TV}(S, \Lambda) := \sup_{A \subset \{0, 1, 2, \dots\}} |\mathbf{P}(S \in A) - \mathbf{P}(\Lambda \in A)| \leq \sum_i p_i^2. \quad (3.19)$$

Proof of Theorem 2.1. Before the proof we introduce some notation. Given X, Y , we generate independent Poisson random variables

$$\eta_i, \quad \xi_{1i}, \quad \xi_{3i}, \quad \xi_{4i}, \quad \Delta_{ri}, \quad i \in T_t, \quad r = 1, 2, 3,$$

with conditional mean values

$$\begin{aligned} \tilde{\mathbf{E}}\eta_i &= \lambda_{it}, & \tilde{\mathbf{E}}\xi_{1i} &= \sum_{j \in T_t^* \setminus \{t\}} p_{ij}, & \tilde{\mathbf{E}}\xi_{3i} &= X_i b_1 \gamma_2, & \tilde{\mathbf{E}}\xi_{4i} &= X_i \bar{b} i^{-1/2}, \\ \tilde{\mathbf{E}}\Delta_{1i} &= \sum_{j \in T_t^* \setminus \{t\}} (\lambda_{ij} - p_{ij}), & \tilde{\mathbf{E}}\Delta_{2i} &= X_i \delta_{2i} i^{-1/2}, & \tilde{\mathbf{E}}\Delta_{3i} &= X_i \delta_{3i} i^{-1/2}. \end{aligned}$$

Here,

$$\delta_{2i} = B_1(T_t^* \setminus \{t\}) - \bar{b}, \quad \delta_{3i} = b_1 \gamma_2 i^{1/2} - \bar{b}, \quad \bar{b} = \min\{B_1(T_t^* \setminus \{t\}), b_1 \gamma_2 i^{1/2}\}.$$

Finally, we define $\xi_{2i} = \xi_{1i} + \Delta_{1i}$, $i \in T_t$ and introduce random variables

$$L_{0t} = \sum_{i \in T_t} \eta_i U_i, \quad L_{rt} = \sum_{i \in T_t} \eta_i \xi_{ri}, \quad r = 1, 2, 3. \quad (3.20)$$

We assume, in addition, that given X, Y , the families of random variables $\{\mathbb{I}_i, i \in T_t\}$ and $\{\xi_{ri}, i \in T_t, r = 1, 2, 3, 4\}$ are conditionally independent, and that $\{\eta_i, i \in T_t\}$ is conditionally independent of the set of edges of $H_{X,Y}$ that are not incident to v_t .

We are ready to start the proof. In view of (3.9) the random variables $d(v_t)$ and L_t have the same asymptotic distribution (if any). We shall prove that L_t converges in distribution to d_* . In the proof we approximate L_t by the random variable L_{3t} (first step of the proof). Afterward we show that L_{3t} converges in distribution to d_* (second step of the proof).

Step 1. In order to show that L_t and L_{3t} have the same asymptotic distribution (if any) we prove the bounds

$$d_{TV}(L_t, L_{0t}) = o(1), \quad d_{TV}(L_{0t}, L_{1t}) = o(1), \quad (3.21)$$

$$\mathbf{E}|L_{1t} - L_{2t}| = o(1), \quad \tilde{L}_{2t} - \tilde{L}_{3t} = o_P(1). \quad (3.22)$$

Here $\tilde{L}_{2t}$ and $\tilde{L}_{3t}$ are marginals of the random vector $(\tilde{L}_{2t}, \tilde{L}_{3t})$ constructed in (3.26) following which has the property that $\tilde{L}_{2t}$ has the same distribution as L_{2t} and $\tilde{L}_{3t}$ has the same distribution as L_{3t} .

Let us prove the first bound of (3.21). We shall show that

$$\tilde{d}_{TV}(L_t, L_{0t}) \mathbb{I}_{A_t} \leq t^{-1} Y_t^2 A_2(T_t). \quad (3.23)$$

From the inequality $\mathbf{E}A_2(T_t) = \sum_{i \in T_t} i^{-1} a_2 \leq c a_2$ we conclude that $A_2(T_t)$ is stochastically bounded. Hence $t^{-1} Y_t^2 A_2(T_t) = o_P(1)$. This bound and (3.23) combined with (3.10) imply

$$\tilde{d}_{TV}(L_t, L_{0t}) \leq \tilde{d}_{TV}(L_t, L_{0t}) \mathbb{I}_{\mathcal{A}_t} + \mathbb{I}_{\overline{\mathcal{A}}_t} = o_P(1).$$

Now the first bound of (3.21) follows from (3.5). It remains to prove (3.23). We set $L'_k = \sum_{i=\lceil at \rceil}^k \mathbb{I}_i U_i + \sum_{i=k+1}^{\lfloor bt \rfloor} \eta_i U_i$. We have, by the triangle inequality,

$$\tilde{d}_{TV}(L_t, L_{0t}) \leq \sum_{k \in T_t} \tilde{d}_{TV}(L'_{k-1}, L'_k).$$

Then we estimate $\tilde{d}_{TV}(L'_{k-1}, L'_k) \leq \tilde{d}_{TV}(\eta_k, \mathbb{I}_k) \leq (kt)^{-1} Y_t^2 X_k^2$. Here the first inequality follows from the properties of the total variation distance. The second inequality follows from Lemma 3.2 and the fact that on the event $\mathcal{A}_t$ we have $p_{kt} = \lambda_{kt}$.

Let us prove the second bound of (3.21). In view of (3.4) it suffices to show that $\tilde{d}_{TV}(L_{0t}, L_{1t}) = o_P(1)$. For this purpose we write, by the triangle inequality,

$$\tilde{d}_{TV}(L_{0t}, L_{1t}) \leq \sum_{k \in T_t} \tilde{d}_{TV}(L_{k-1}^*, L_k^*), \quad (3.24)$$

where $L_k^* := \sum_{i=\lceil at \rceil}^k \eta_i U_i + \sum_{i=k+1}^{\lfloor bt \rfloor} \eta_i \xi_{1i}$, and estimate

$$\tilde{d}_{TV}(L_{k-1}^*, L_k^*) \leq \tilde{d}_{TV}(\eta_k U_k, \eta_k \xi_{1k}) \leq \tilde{\mathbf{P}}(\eta_k \neq 0) \tilde{d}_{TV}(U_k, \xi_{1k}). \quad (3.25)$$

Now, invoking the inequalities

$$\tilde{\mathbf{P}}(\eta_k \neq 0) = 1 - e^{-\lambda_{kt}} \leq \lambda_{kt} \quad \text{and} \quad \tilde{d}_{TV}(U_k, \xi_{1k}) \leq \sum_{j \in T_k^* \setminus \{t\}} p_{kj}^2,$$

see (3.19), we obtain from (3.24), (3.25) and (3.11) that

$$\tilde{d}_{TV}(L_{0t}, L_{1t}) \leq Q_{XY}(t) = o_P(1).$$

Let us prove the first bound of (3.22). We observe that

$$|L_{2t} - L_{1t}| = L_{2t} - L_{1t} = \sum_{i \in T_t} \eta_i \Delta_{1i}$$

and

$$\tilde{\mathbf{E}} \sum_{i \in T_t} \eta_i \Delta_{1i} = \sum_{i \in T_t} \lambda_{it} \sum_{j \in T_t^* \setminus \{t\}} (\lambda_{ij} - 1) \mathbb{I}_{\{\lambda_{ij} > 1\}} \leq Q_{XY}(t).$$

We obtain $\mathbf{E}|L_{2t} - L_{1t}| \leq \mathbf{E}Q_{XY}(t) = o(1)$, see (3.11).

Let us prove the second bound of (3.22). We note that the random vector

$$(\tilde{L}_{2t}, \tilde{L}_{3t}), \quad \tilde{L}_{2t} = \sum_{i \in T_t} \eta_i (\xi_{4i} + \Delta_{2i}), \quad \tilde{L}_{3t} = \sum_{i \in T_t} \eta_i (\xi_{4i} + \Delta_{3i}) \quad (3.26)$$

has the marginal distributions of (L_{2t}, L_{3t}) . In addition, since $\Delta_{2i}, \Delta_{3i} \geq 0$ and at most one of them is nonzero, we have $|\Delta_{2i} - \Delta_{3i}| = \Delta_{2i} + \Delta_{3i}$. Therefore, we can write

$$\tilde{\Delta} := |\tilde{L}_{2t} - \tilde{L}_{3t}| \leq \sum_{i \in T_t} |\eta_i| |\Delta_{2i} - \Delta_{3i}| = \sum_{i \in T_t} \eta_i (\Delta_{2i} + \Delta_{3i}). \quad (3.27)$$

We remark that given X, Y , the random variable $\Delta_{2i} + \Delta_{3i}$ has Poisson distribution with (conditional) mean value

$$\tilde{\mathbf{E}}(\Delta_{2i} + \Delta_{3i}) = X_i t^{-1/2} \delta_i, \quad \delta_i := |B_1(T_i^* \setminus \{t\}) - b_1 \gamma_2 i^{1/2}|.$$

Therefore, (3.27) implies $\tilde{\mathbf{E}}\tilde{\Delta} \leq t^{-1/2} Y_t \sum_{i \in T_t} X_i^2 i^{-1} \delta_i$. Next, for $0 < \varepsilon < 1$, we have

$$\mathbf{E}\mathbb{I}_{\mathcal{B}_t(\varepsilon)} \tilde{\Delta} \leq \mathbf{E}\mathbb{I}_{\mathcal{B}_t(\varepsilon)} t^{-1/2} Y_t \sum_{i \in T_t} X_i^2 i^{-1} \delta_i = b_1 a_2 t^{-1/2} \sum_{i \in T_t} i^{-1} \mathbf{E} \delta_i \mathbb{I}_{\mathcal{B}_t(\varepsilon)}.$$

Invoking upper bound (3.13) for $\mathbf{E} \delta_i \mathbb{I}_{\mathcal{B}_t(\varepsilon)}$, we obtain $\mathbf{E}\mathbb{I}_{\mathcal{B}_t(\varepsilon)} \tilde{\Delta} \leq c b_1^{3/2} a_2 \varepsilon + o(1)$. Finally, this bound combined with Markov's inequality and (3.7) yields

$$\mathbf{P}(\tilde{\Delta} \geq 1) = \mathbf{P}(\{\tilde{\Delta} \geq 1\} \cap \mathcal{B}_t(\varepsilon)) + o(1) \leq \mathbf{E}\mathbb{I}_{\mathcal{B}_t(\varepsilon)} \tilde{\Delta} + o(1) \leq c b_1^{3/2} a_2 \varepsilon + o(1).$$

We conclude that $\mathbf{P}(\tilde{\Delta} \neq 0) = \mathbf{P}(\tilde{\Delta} \geq 1) = o(1)$, thus obtaining the second bound of (3.22).

Step 2. Here we prove that L_{3t} converges in distribution to d_* . Before the proof, we introduce some notation. Let Y_* be a random variable with the same distribution as Y_1 and independent of X, Y . Given X, Y, Y_* , we generate independent Poisson random variables $\eta_k^*, k \in T_t$ with (conditional) mean values $\mathbf{E}(\eta_k^* | X, Y, Y_*) = \lambda_{k*}$, where $\lambda_{k*} = X_k Y_*(kt)^{-1/2}$. We assume that, given X, Y, Y_* , the family of random variables $\{\eta_k^*, k \in T_t\}$ is conditionally independent of $\{\xi_{3k}, k \in T_t\}$. Define $L_t^* = \sum_{k \in T_t} \eta_k^* \xi_{3k}$, and let d_* be defined in the same way as d_* , but with λ_1 replaced by $\lambda_* = Y_* a_1 \gamma_1$. Denote

$$f_{\mathcal{X}}(z) = \mathbf{E} e^{iz\mathcal{X}_1}, \quad \bar{f}_{\mathcal{X}}(z) = \sum_{r \geq 0} e^{izr} \bar{p}_r, \quad \bar{p}_r = \bar{\lambda}^{-1} \sum_{k \in T_t} \lambda_{k*} \mathbb{I}_{\{\xi_{3k}=r\}}, \quad \bar{\lambda} = \sum_{k \in T_t} \lambda_{k*},$$

$$\delta = (\bar{f}_{\mathcal{X}}(z) - 1) \bar{\lambda} - (f_{\mathcal{X}}(z) - 1) \lambda_*, \quad f(z) = \bar{\mathbf{E}} e^{izd_*}, \quad \bar{f}(z) = \bar{\mathbf{E}} e^{izL_t^*}.$$

Here and following, i denotes the imaginary unit, z is a real number. $\bar{\mathbf{E}}$ denotes the conditional expectation given X, Y, Y_* and $\{\xi_{3k}, k \in T_t\}$. By $\mathbf{E}_*$ we denote the conditional expectation given Y_* . We recall that random variables $\mathcal{X}_1, \mathcal{X}_2, \dots$ are defined in the statement of Theorem 2.1. Given $\varepsilon > 0$, introduce the event $\mathcal{D} = \mathcal{D}_\varepsilon = \{|A_1(T_t) - \gamma_1 a_1 t^{1/2}| < \varepsilon t^{1/2} \min\{1, \gamma_1 a_1\}\}$ and let $\bar{\mathcal{D}}$ denote the complement event. Furthermore, select the number $T = T_\varepsilon > 1/\varepsilon$ such that $\mathbf{P}(\mathcal{X}_1 \geq T) < \varepsilon$.

We note that L_t^* is defined in the same way as L_{3t} above, but with Y_t replaced by Y_* . Hence L_{3t} has the same distribution as L_t^* . Similarly, d_* has the same distribution as d_* . Therefore, it suffices to show that L_t^* converges in distribution to d_* . For this purpose, we show the convergence of Fourier–Stieltjes transforms $\mathbf{E} e^{izL_t^*} \rightarrow \mathbf{E} e^{izd_*}$. We observe that, given Y_* , the conditional distribution of d_* is the compound Poisson distribution with the characteristic function $f(z) = e^{\lambda_*(f_{\mathcal{X}}(z)-1)}$. Similarly, given X, Y, Y_* and $\{\xi_{3k}, k \in T_t\}$, the conditional distribution of L_t^* is the compound Poisson distribution with the characteristic function $\bar{f}(z) = e^{\bar{\lambda}(\bar{f}_{\mathcal{X}}(z)-1)}$. In the proof we exploit the convergence $\bar{\lambda} \rightarrow \lambda_*$ and $\bar{f}_{\mathcal{X}}(z) \rightarrow$

$f_{\varkappa}(z)$. Denote $\Delta^*(z) = e^{izL_t^*} - e^{izd_*}$. We shall show below that, for any real z and any realized value Y_* there exists a positive constant $c^* = c^*(z, Y_*)$ such that for every $0 < \varepsilon < 0.5$ we have

$$\limsup_t |\mathbf{E}(\Delta^*(z)|Y_*)| < c^* \varepsilon. \quad (3.28)$$

We remark that (3.28) implies $\mathbf{E}(\Delta^*(z)|Y_*) = o(1)$. This fact together with the simple inequality $|\Delta^*(z)| \leq 2$ yields $\mathbf{E}\Delta^*(z) = o(1)$, by Lebesgue's dominated convergence theorem. Finally, the identity $\mathbf{E}\Delta^*(z) = \mathbf{E}e^{izL_t^*} - \mathbf{E}e^{izd_*}$ implies the desired convergence $\mathbf{E}e^{izL_t^*} \rightarrow \mathbf{E}e^{izd_*}$ as $t \rightarrow +\infty$.

Let us fix $0 < \varepsilon < 0.5$ and prove (3.28). We write

$$\mathbf{E}_* \Delta^*(z) = I_1 + I_2, \quad I_1 = \mathbf{E}_* \Delta^*(z) \mathbb{I}_{\mathcal{D}}, \quad I_2 = \mathbf{E}_* \Delta^*(z) \mathbb{I}_{\overline{\mathcal{D}}}, \quad (3.29)$$

and observe that $|I_2| \leq 2\mathbf{P}_*(\overline{\mathcal{D}}) = 2\mathbf{P}(\overline{\mathcal{D}}) = o(1)$. Indeed, the bound $\mathbf{P}(\overline{\mathcal{D}}) = o(1)$ follows from (3.14), by Markov's inequality. Let us estimate I_1 . Combining the identity $\bar{\mathbf{E}}\Delta^*(z) = \bar{\mathbf{E}}f(z)(e^\delta - 1)$ with the inequalities $|f(z)| \leq 1$ and $|e^\delta - 1| \leq |\delta|e^{|\delta|}$, we obtain

$$|I_1| \leq \mathbf{E}_* |\delta| e^{|\delta|} \mathbb{I}_{\mathcal{D}} \leq c_1^* \mathbf{E}_* |\delta| \mathbb{I}_{\mathcal{D}}. \quad (3.30)$$

Here we estimated $e^{|\delta|} \leq e^{6\lambda_*} =: c_1^*$ using the inequalities

$$|\delta| \leq 2\bar{\lambda} + 2\lambda_*, \quad \bar{\lambda} = Y_* t^{-1/2} A_1(T_t) \leq 2\lambda_*.$$

We remark that the last inequality holds, provided that event $\mathcal{D}$ occurs. Here and in what follows $c_1^*, c_2^*, \dots$ denote positive numbers that do not depend on t .

We complete the proof of (3.28) by showing that $\mathbf{E}_* |\delta| \mathbb{I}_{\mathcal{D}} \leq (c_2^* + c_3^* \lambda_* + c_4^* \lambda_*) \varepsilon + o(1)$, see also (3.29), (3.30). To this aim we write

$$\delta = (\tilde{f}_{\varkappa}(z) - 1)(\bar{\lambda} - \lambda_*) + (\tilde{f}_{\varkappa}(z) - f_{\varkappa}(z))\lambda_*$$

and estimate $|\delta| \leq 2|\bar{\lambda} - \lambda_*| + \lambda_* |\tilde{f}_{\varkappa}(z) - f_{\varkappa}(z)|$. The inequality, which holds on the event $\mathcal{D}$, $|\bar{\lambda} - \lambda_*| \leq Y_* \varepsilon$ implies $\mathbf{E}_* |\bar{\lambda} - \lambda_*| \mathbb{I}_{\mathcal{D}} \leq c_2^* \varepsilon$ with $c_2^* := Y_*$. It remains to show that

$$\mathbf{E}_* |\tilde{f}_{\varkappa}(z) - f_{\varkappa}(z)| \mathbb{I}_{\mathcal{D}} \leq (c_3^* + c_4^*) \varepsilon + o(1).$$

We split

$$\tilde{f}_{\varkappa}(z) - f_{\varkappa}(z) = \sum_{r \geq 0} e^{izr} (\bar{p}_r - p_r) = R_1 - R_2 + R_3,$$

and estimate separately the terms

$$R_1 = \sum_{r \geq T} e^{izr} \bar{p}_r, \quad R_2 = \sum_{r \geq T} e^{izr} p_r, \quad R_3 = \sum_{0 \leq r < T} e^{izr} (\bar{p}_r - p_r).$$

Here we denote $p_r = \mathbf{P}(\varkappa_1 = r)$.

The upper bound for R_2 follows by the choice of T

$$|R_2| \leq \sum_{r \geq T} p_r = \mathbf{P}(\varkappa_1 \geq T) < \varepsilon.$$

Next, combining the identities $\bar{p}_r = (A_1(T_t))^{-1} \sum_{k \in T_t} k^{-1/2} X_k \mathbb{I}_{\{\xi_{3k}=r\}}$ and

$$\sum_{r \geq T} \sum_{k \in T_t} k^{-1/2} X_k \mathbb{I}_{\{\xi_{3k}=r\}} = \sum_{k \in T_t} k^{-1/2} X_k \mathbb{I}_{\{\xi_{3k} \geq T\}}$$

with the inequality $A_1(T_t) \geq t^{1/2} a_1 \gamma_1 / 2$, which holds on the event $\mathcal{D}$, we obtain

$$|R_1|_{\mathbb{I}_{\mathcal{D}}} \leq \sum_{r \geq T} \bar{p}_r \leq \frac{2}{a_1 \gamma_1 t^{1/2}} \sum_{k \in T_t} \frac{X_k}{k^{1/2}} \mathbb{I}_{\{\xi_{3k} \geq T\}} \leq \frac{2}{a_1 \gamma_1 t^{1/2}} \sum_{k \in T_t} \frac{X_k \xi_{3k}}{T k^{1/2}}.$$

Now, the identity $\mathbf{E}_* X_k \xi_{3k} = a_2 b_1 \gamma_2$ implies $\mathbf{E}_* |R_1|_{\mathbb{I}_{\mathcal{D}}} \leq c_4^* T^{-1} \leq c_4^* \varepsilon$.

Let us estimate R_3 . We denote $p'_r = A_1(T_t)(a_1 \gamma_1 t^{1/2})^{-1} \bar{p}_r$ and observe that the inequality $|A_1(T_t)(a_1 \gamma_1 t^{1/2})^{-1} - 1| \leq \varepsilon$, which holds on the event $\mathcal{D}$, implies

$$\left| \sum_{0 \leq r \leq T} e^{itr} (\bar{p}_r - p'_r) \right|_{\mathbb{I}_{\mathcal{D}}} \leq \varepsilon \sum_{0 \leq r \leq T} \bar{p}_r \leq \varepsilon.$$

In the last inequality we use the fact that the probabilities $\{\bar{p}_r\}_{r \geq 0}$ sum to 1. It follows now that

$$|R_3|_{\mathbb{I}_{\mathcal{D}}} \leq \varepsilon + \sum_{0 \leq r \leq T} |p'_r - p_r|. \quad (3.31)$$

Next we estimate

$$\mathbf{E}_* |p'_r - p_r| \leq \mathbf{E}_* |p'_r - \mathbf{E}_* p'_r| + |\mathbf{E}_* p'_r - p_r| \quad (3.32)$$

and

$$(\mathbf{E}_* |p'_r - \mathbf{E}_* p'_r|)^2 \leq \mathbf{E}_* |p'_r - \mathbf{E}_* p'_r|^2 \leq (a_1 \gamma_1 t^{1/2})^{-2} \sum_{k \in T_t} k^{-1} a_2 \leq c t^{-1} a_2 a_1^{-2}, \quad (3.33)$$

$$|\mathbf{E}_* p'_r - p_r| = p_r |1 - (\gamma_1 t^{1/2})^{-1} \sum_{k \in T_t} k^{-1/2}| \leq c t^{-1}. \quad (3.34)$$

In (3.33) we first apply the Cauchy–Schwartz inequality, then the linearity of variance of a sum of independent random variables, and then the inequality $\mathbf{Var} X_k \mathbb{I}_{\{\xi_{3k}=r\}} \leq \mathbf{E}(X_k \mathbb{I}_{\{\xi_{3k}=r\}})^2 \leq a_2$. In (3.34) we use the identity $\mathbf{E}_* X_k \mathbb{I}_{\{\xi_{3k}=r\}} = a_1 p_r$ and (3.6). From (3.32), (3.33), (3.34) we conclude that $\mathbf{E}_* |p'_r - p_r| = O(t^{-1/2})$. Finally, (3.31) implies

$$\mathbf{E}_* |R_3|_{\mathbb{I}_{\mathcal{D}}} \leq \varepsilon + O(|T| t^{-1/2}) = \varepsilon + o(1).$$

□

3.2. Proof of Theorem 2.2

Here, we assume that $\tau(t) = t^\nu$, $\nu > 1$. In the proof, we apply the following simple approximations

$$\sum_{k \in T_t} k^{-(1-2\nu)/(2\nu)} = t^{1/2} \gamma'_1 + r t^{(2\nu)^{-1}-1}, \quad \sum_{j \in T_k^*} j^{-1/2} = k^{(2\nu)^{-1}} \gamma'_2 + r' k^{-(2\nu)^{-1}}, \quad (3.35)$$

$$\gamma'_1 := 2\nu(b^{(2\nu)^{-1}} - a^{(2\nu)^{-1}}), \quad \gamma'_2 := 2(a^{-(2\nu)^{-1}} - b^{-(2\nu)^{-1}}),$$

where $|r|, |r'| \leq c$. We also make use of relations (3.7), (3.9), (3.10) and (3.11), which remain valid in the case where $\tau(t) = t^\nu$, and of the inequalities, for $k \in T_t$,

$$|\mathbf{E}B_1(T_k^* \setminus \{t\}) - b_1 \gamma'_2 k^{1/(2\nu)}| \leq c b_1 k^{-1/(2\nu)}, \quad (3.36)$$

$$\mathbf{E}|B_1(T_k^* \setminus \{t\}) - b_1 \gamma'_2 k^{1/(2\nu)}| \mathbb{I}_{\mathcal{B}_t(\varepsilon)} \leq c k^{1/(2\nu)} (\varepsilon b_1^{1/2} + \mathbf{E}Y_1 \mathbb{I}_{\{Y_1 > \varepsilon t_-\}}) + c b_1 k^{-1/(2\nu)}. \quad (3.37)$$

We note that (3.36) follows from the second identity of (3.35), and (3.37) is obtained in the same way as (3.13).

Proof of Theorem 2.2. Before the proof we introduce some notation. Given $\varepsilon \in (0, 1)$, denote

$$\zeta = \sum_{k \in T_t} \lambda_{kt} \zeta_k, \quad \zeta_k = \beta_k b_1 X_k \mathbb{I}'_k, \quad \mathbb{I}'_k = \mathbb{I}_{\{\beta_k b_1 X_k < \varepsilon\}}, \quad \beta_k = k^{(1-\nu)/(2\nu)} \gamma'_2.$$

Given X, Y , we generate independent Poisson random variables $\eta_k, \hat{\xi}_{3k}$, $k \in T_t$, with (conditional) mean values $\tilde{\mathbf{E}}\eta_k = \lambda_{kt}$, $\tilde{\mathbf{E}}\hat{\xi}_{3k} = \beta_k b_1 X_k$. We also generate independent Bernoulli random variables $\tilde{\mathbb{I}}_k$, $k \in T_t$ with success probabilities

$$\tilde{\mathbf{P}}(\tilde{\mathbb{I}}_k = 1) = 1 - \tilde{\mathbf{P}}(\tilde{\mathbb{I}}_k = 0) = \zeta_k$$

and introduce random variables

$$\hat{L}_{3t} = \sum_{k \in T_t} \eta_k \hat{\xi}_{3k}, \quad L_{4t} = \sum_{k \in T_t} \mathbb{I}_k \hat{\xi}_{3k}, \quad L_{5t} = \sum_{k \in T_t} \mathbb{I}_k \mathbb{I}'_k \hat{\xi}_{3k}, \quad L_{6t} = \sum_{k \in T_t} \mathbb{I}_k \tilde{\mathbb{I}}_k.$$

We assume that, given X, Y , the sequences $\{\mathbb{I}_k, k \in T_t\}$, $\{\tilde{\mathbb{I}}_k, k \in T_t\}$, $\{\eta_k, k \in T_t\}$, $\{\hat{\xi}_{3k}, k \in T_t\}$ are conditionally independent. Furthermore, we define the random variable L_{7t} as follows. We first generate X, Y . Then, given X, Y , we generate a Poisson random variable with the conditional mean value ζ . The realized value of the Poisson random variable is denoted L_{7t} . Thus, we have $\mathbf{P}(L_{7t} = r) = \mathbf{E}e^{-\zeta} \zeta^r / r!$, for $r = 0, 1, \dots$.

Now we are ready to prove Theorem 2.2. Proceeding in the same way as in the proof of (3.21), (3.22) above (and making use of (3.7), (3.9), (3.10), (3.11), (3.36), (3.37)) we show that the random variables $d(v_t)$ and $\hat{L}_{3t}$ have the same asymptotic distribution (if any). It remains to show that $\hat{L}_{3t}$ converges in distribution to Λ_3 . For this purpose we prove that

$$d_{TV}(\hat{L}_{3t}, L_{4t}) = o(1), \quad \mathbf{E}(L_{4t} - L_{5t}) = o(1), \quad d_{TV}(L_{6t}, L_{7t}) = o(1) \quad (3.38)$$

and there exists $c > 0$, depending only on a, b, ν , such that

$$d_{TV}(L_{5t}, L_{6t}) \leq ca_2 b_1^2 \varepsilon. \quad (3.39)$$

Furthermore, for every real z ,

$$\mathbf{E}e^{izL_{7t}} - \mathbf{E}e^{iz\Lambda_3} = o(1) \quad (3.40)$$

(here i denotes the imaginary unit). Clearly, from (3.38), (3.40), (3.39) we obtain that $\hat{L}_{3t}$ converges in distribution to Λ_3 .

Let us prove (3.38), (3.39), (3.40). The first bound of (3.38) is obtained in the same way as that of (3.21). To show the second bound of (3.38) we write

$$\tilde{\mathbf{E}}(L_{4t} - L_{5t}) = \sum_{k \in T_t} (1 - \mathbb{I}'_k) \tilde{\mathbf{E}} \mathbb{I}_k \tilde{\mathbf{E}} \hat{\xi}_{3k} = Y_t b_1 t^{-1/2} \sum_{k \in T_t} (1 - \mathbb{I}'_k) X_k^2 \beta_k k^{-1/2}.$$

and apply the inequality

$$\mathbf{E}X_k^2(1 - \mathbb{I}'_k) \leq \mathbf{E}X_{\underline{t}}^2(1 - \mathbb{I}'_{\underline{t}}), \quad k \in T_t. \quad (3.41)$$

where $\underline{t} = \min\{k : k \in T_t\}$. In this way we obtain

$$\mathbf{E}(L_{4t} - L_{5t}) = \mathbf{E}\tilde{\mathbf{E}}(L_{4t} - L_{5t}) \leq S_t b_1^2 \mathbf{E}X_{\underline{t}}^2(1 - \mathbb{I}'_{\underline{t}}) = o(1).$$

Here $S_t = t^{-1/2} \sum_{k \in T_t} \beta_k k^{-1/2}$ is uniformly bounded, and $\mathbf{E}X_{\underline{t}}^2(1 - \mathbb{I}'_{\underline{t}}) = o(1)$, since $\underline{t} \rightarrow +\infty$ as $t \rightarrow +\infty$.

Let us prove the third bound of (3.38). In view of (3.4) it suffices to show that $\tilde{d}_{TV}(L_{6t}, L_{7t}) = o_P(1)$. For this purpose we write

$$\tilde{d}_{TV}(L_{6t}, L_{7t}) \leq \mathbb{I}_{\mathcal{A}_t} \tilde{d}_{TV}(L_{6t}, L_{7t}) + \mathbb{I}_{\bar{\mathcal{A}}_t},$$

where $\mathbb{I}_{\bar{\mathcal{A}}_t} = o_P(1)$, see (3.10), and estimate using Le Cam's inequality (3.19)

$$\mathbb{I}_{\mathcal{A}_t} \tilde{d}_{TV}(L_{6t}, L_{7t}) \leq \mathbb{I}_{\mathcal{A}_t} \sum_{k \in T_t} \tilde{\mathbf{P}}^2(\mathbb{I}_k \tilde{\mathbb{I}}_k = 1) \mathbb{I}'_k \leq Y_t^2 b_1^2 t^{-1} \sum_{k \in T_t} k^{-1} \beta_k^2 X_k^4 = o_P(1)$$

Here we used the simple inequality $t^{-1} \sum_{k \in T_t} k^{-1} \beta_k^2 X_k^4 \leq ct^{-2\nu} \sum_{k \leq bt^\nu} X_k^4$ and the fact that $\mathbf{E}X_1^2 < \infty$ implies the bound $n^{-2} \sum_{k \leq n} X_k^4 = o_P(1)$, as $n \rightarrow +\infty$.

Let us prove (3.39). Proceeding as in (3.24), (3.25) and using the identity $\tilde{\mathbb{I}}_k = \tilde{\mathbb{I}}_k \mathbb{I}'_k$ we have

$$\tilde{d}_{TV}(L_{5t}, L_{6t}) \leq \sum_{k \in T_t} \mathbb{I}'_k \tilde{\mathbf{P}}(\mathbb{I}_k \neq 0) \tilde{d}_{TV}(\hat{\xi}_{3k}, \tilde{\mathbb{I}}_k).$$

Next, we estimate $\mathbb{I}'_k \tilde{d}_{TV}(\hat{\xi}_{3k}, \tilde{\mathbb{I}}_k) \leq \zeta_k^2$, by Le Cam's inequality (3.19), and invoke the inequality $\tilde{\mathbf{P}}(\mathbb{I}_k \neq 0) \leq \lambda_{kt}$. We obtain

$$\tilde{d}_{TV}(L_{5t}, L_{6t}) \leq \sum_{k \in T_t} \mathbb{I}'_k \lambda_{kt} \zeta_k^2 \leq \varepsilon \sum_{k \in T_t} \lambda_{kt} \zeta_k.$$

Here we estimated $\zeta_k^2 \leq \varepsilon \zeta_k$. Now, the inequalities

$$d_{TV}(L_{5t}, L_{6t}) \leq \tilde{\mathbf{E}} d_{TV}(L_{5t}, L_{6t}) \leq \varepsilon \sum_{k \in T_t} \mathbf{E} \lambda_{kt} \zeta_k \leq a_2 b_1^2 S_t \varepsilon$$

and the fact that S_t is uniformly bounded, i.e., $S_t \leq c$, for some $c > 0$, imply (3.39).

Finally, we show (3.40). We write $\tilde{\mathbf{E}} e^{izL_{7t}} = e^{\zeta(e^{iz}-1)}$ and use the bound

$$Y_t b_1 a_2 \gamma - \zeta = o_P(1). \quad (3.42)$$

We note that since the function $u \rightarrow e^{u(e^{iz}-1)}$ is bounded and uniformly continuous, for $u \geq 0$, relation (3.42) implies the convergence

$$\mathbf{E} e^{izL_{7t}} = \mathbf{E} e^{\zeta(e^{iz}-1)} \rightarrow \mathbf{E} e^{Y_t b_1 a_2 \gamma (e^{iz}-1)} = \mathbf{E} e^{iz\Lambda_3}.$$

It remains to prove (3.42). Proceeding as in the proof of the second bound of (3.38) we obtain

$$\zeta = Y_t b_1 \gamma_2' t^{-1/2} \sum_{k \in T_t} X_k^2 k^{(2v)^{-1}-1} + o_P(1). \quad (3.43)$$

Then we split $\gamma = \gamma_1' \gamma_2'$ and invoke the expression for γ_1' obtained from (3.35). We obtain

$$Y_t b_1 a_2 \gamma = Y_t b_1 \gamma_2' t^{-1/2} \sum_{k \in T_t} k^{-(1-2v)/(2v)} a_2 + o_P(1). \quad (3.44)$$

Now, we observe that (3.42) follows from (3.43), (3.44) and the bound

$$t^{-1/2} \sum_{k \in T_t} (a_2 - X_k^2) k^{(2v)^{-1}-1} = o_P(1), \quad (3.45)$$

which is obtained from Chebyshev's inequality using a truncation argument. $\square$

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