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LONG PATHS IN RANDOM APOLLONIAN NETWORKS

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Abstract We consider the length $L(n)$ of the longest path in a randomly generated Apollonian Network (ApN) $\mathcal{A}_n$. We show that with high probability $L(n) \leq ne^{-\log^c n}$ for any constant $c < 2/3$.

1. INTRODUCTION

This article concerns the length of the longest path in a random Apollonian Network (ApN) $\mathcal{A}_n$. We start with a triangle $T_0 = xyz$ in the plane. We then place a point v_1 in the centre of this triangle, creating three triangular faces. We choose one of these faces at random and place a point v_2 in its middle. There are now five triangular faces. We choose one at random and place a point v_3 in its center. In general, after we have added $v_1, v_2, \ldots, v_1$, there will be $2n + 1$ triangular faces. We choose one at random and place v_n inside it. The random graph $\mathcal{A}_n$ is the graph induced by this embedding. It has $n + 3$ vertices and $3n + 6$ edges.

This graph has been the object of study recently; in the context of scale-free graphs [5]. Properties of its degree sequence, properties of the spectra of its adjacency matrix, and its diameter were determined. The diameter result was improved, and the diameter was determined asymptotically [3, 4]. The study [4] proves the following result concerning the length of the longest path in $\mathcal{A}_n$:

Theorem 1.1. *There exists an absolute constant α such that if $L(n)$ denotes the length of the longest path in $\mathcal{A}_n$ then*

$$\mathbf{Pr}\left(L(n) \geq \frac{n}{\log^\alpha n}\right) \leq \frac{1}{\log^\alpha n}.$$

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The value of α from [4] is rather small, and we will assume for the purposes of this proof that

$$\alpha < \frac{1}{3}. \tag{1.1}$$

The aim of this work is to give the following improvement on Theorem 1.1:

Theorem 1.2.

$$\mathbf{Pr}\left(L(n) \geq ne^{-\log^c n}\right) \leq O\left(e^{-\log^{c/2} n}\right)$$

for any constant $c < 2/3$.

This is most likely far from the truth. It is reasonable to conjecture that, in fact, $L(n) \leq n^{1-\varepsilon}$ w.h.p. for some positive $\varepsilon > 0$. For lower bounds, [4] shows that $L(n) \geq n^{\log_3 2} + 2$ always and $\mathbf{E}(L(n)) = \Omega(n^{0.8})$. An $\Omega(n^{\log_3 2})$ lower bound for arbitrary 3-connected planar graphs has been proven [1].

2. OUTLINE PROOF STRATEGY

We take an arbitrary path P in $\mathcal{A}_n$ and bound its length as follows. We add vertices to the interior of xyz in rounds. In round i we add σ_i vertices. We start with $\sigma_0 = n^{1/2}$ and choose $\sigma_i \gg \sigma_{i-1}$, where $A \gg B$ i.f.f $B = o(A)$. We will argue inductively that P visits only $\tau_{i-1} = o(\sigma_{i-1})$ faces of $\mathcal{A}_{\sigma_{i-1}}$ and then use Lemma 4.1 to argue that roughly a fraction τ_{i-1}/σ_{i-1} of the σ_i new vertices go into faces visited by P. We then use a variant (Lemma 5.1) of Theorem 1.1 to argue that w.h.p. $\frac{\tau_i}{\sigma_i} \leq \frac{\tau_{i-1}}{2\sigma_{i-1}}$. Theorem 1.2 will follow easily from this.

3. PATHS AND TRIANGLES

Fix $1 \leq \sigma \leq n$ and let $\mathcal{A}_\sigma$ denote the ApN we have after inserting σ vertices A interior to T_0. It has $2\sigma + 1$ faces, which we denote by $\mathcal{T} = \{T_1, T_2, \ldots, T_{2\sigma+1}\}$. Now add N more vertices B to create a larger network $\mathcal{A}_{\sigma'}$, where $\sigma' = \sigma + N$. Now consider a path $P = x_1, x_2, \ldots, x_m$ through $\mathcal{A}_{\sigma'}$. Let $I = \{i : x_i \in A\} = \{i_1, i_2, \ldots, i_\tau\}$. Note that $Q = (i_1, i_2, \ldots, i_\tau)$ is a path of length $\tau - 1$ in $\mathcal{A}_\sigma$. This is because $i_k i_{k+1}$, $1 \leq k < \tau$ must be an edge of some face in $\mathcal{T}$. We also see that for any $1 \leq k < \tau$, the vertices x_j, $i_k < j < i_{k+1}$ will all be interior to the same face T_l for some $l \in [2\sigma + 1]$.

We summarize this in the following lemma, using the notation of the preceding paragraph.

Lemma 3.1. *Suppose that $1 \leq \sigma < \sigma' \leq n$ and that Q is a path of $\mathcal{A}_\sigma$ that is obtained from a path P in $\mathcal{A}_{\sigma'}$ by omitting the vertices in B.*

Suppose that Q has τ vertices and that P visits the interior of τ' faces from $\mathcal{T}$. Then

$$\tau - 1 \le \tau' \le \tau + 1.$$

Proof. The path P breaks into vertices of $\mathcal{A}_\sigma$ plus $\tau + 1$ intervals where, in an interval, it visits the interior of a single face in $\mathcal{T}$. This justifies the upper bound. The lower bound comes from the fact that except for the face in which it starts, if P reenters a face xyz, then it cannot leave it, because it will have already visited all three vertices x, y, z. Thus, at most two of the aforementioned intervals can represent a repeated face. □

4. A STRUCTURAL LEMMA

Let

$$\lambda_1 = \log^2 n.$$

A sequence of events $\mathcal{E}_n$ holds *quite surely* (q.s.) if $\mathbf{Pr}(\neg\mathcal{E}_n) = O(n^{-K})$ for any constant $K > 0$.

Lemma 4.1. *The following holds for all i. Let $\sigma = \sigma_i$ and suppose that $\lambda_1 \le \tau \ll \sigma$. Suppose that $T_1, T_2, \ldots, T_\tau$ is a set of triangular faces of $\mathcal{A}_\sigma$. Suppose that $N \gg \sigma$ and that when adding N vertices to $\mathcal{A}_\sigma$ we find that M_j vertices are placed in T_j for $j = 1, 2, \ldots, \tau$. Then, for all $J \subseteq [2\sigma + 1]$, $|J| = \tau$, we have*

$$\sum_{j \in J} M_j \le \frac{100\tau N}{\sigma} \log\left(\frac{\sigma}{\tau}\right).$$

This holds q.s. for all choices of τ, σ, and $T_1, T_2, \ldots, T_\tau$.

Proof. We consider the following process that starts with s newly *born* particles. Once a particle is born, it waits an exponentially mean one distributed amount of time. After this time, it simultaneously *dies* and gives birth to k new particles, and so on. A birth corresponds to a vertex of our network and a particle corresponds to a face.

Let Z_t denote the number of deaths up to time t. The number of particles in the system is $\beta_N = s + N(k - 1)$. Then we have

$$\mathbf{Pr}(Z_{t+dt} = N) = \beta_{N-1}\,\mathbf{Pr}(Z_t = N - 1)dt + (1 - \beta_N dt)\,\mathbf{Pr}(Z_t = N).$$

So, if $p_N(t) = \mathbf{Pr}(Z_t = N)$, we have $f_N(0) = 1_{N=s}$ and

$$p_N'(t) = \beta_{N-1} p_{N-1}(t) - \beta_N p_N(t).$$

This yields

$$\begin{aligned} p_N(t) &= \prod_{i=1}^{N} \frac{(k-1)(i-1)+s}{(k-1)i} \quad \times e^{-st}(1-e^{-(k-1)t})^N \\ &= A_{k,N,s} e^{-st}(1-e^{-(k-1)t})^N. \end{aligned}$$

We have $A_{3,0,s} = 1$. When s is even, $s, N \to \infty$, and $k = 3$, we have

$$\begin{aligned} A_{3,N,s} &= \prod_{i=1}^{N} \left(\frac{s/2+i-1}{i} \right) = \binom{N+s/2-1}{s/2-1} \\ &\approx \left(1+\frac{s-2}{2N}\right)^N \left(1+\frac{2N}{s-2}\right)^{s/2-1} \sqrt{\frac{2N+s}{2\pi N s}}. \end{aligned}$$

We also need to have an upper bound for a small, even s, $N^2 = o(s)$, say. In this case, we use

$$A_{3,N,s} \le s^N.$$

When $s \ge 3$ is odd, $s, N \to \infty$ (no need to deal with small N here), and $k = 3$ we have

$$\begin{aligned} A_{3,N,s} &= \prod_{i=1}^{N} \left(\frac{2i-2+s}{2i} \right) = \frac{(s-1+2N)!((s-1)/2)!}{2^{2N}(s-1)!N!((s-1)/2+N)!} \\ &\approx \left(1+\frac{s-1}{2N}\right)^N \left(1+\frac{2N}{s-1}\right)^{(s-1)/2} \frac{1}{(2\pi N)^{1/2}}. \end{aligned}$$

We now consider $\tau \to \infty, \tau \ll \sigma, N \ge m \ge 2\tau N/\sigma \gg \tau$ and arbitrary t (under the assumption that τ is odd and σ is odd). (We sometimes use $A \le_b B$ in place of $A = O(B)$).

$$\begin{aligned} &\mathbf{Pr}(M_1 + \cdots + M_\tau = m \mid M_1 + \cdots + M_\sigma = N) \\ &= \frac{\mathbf{Pr}(M_1 + \cdots + M_\tau = m)\,\mathbf{Pr}(M_{\tau+1} + \cdots + M_\sigma = N-m)}{\mathbf{Pr}(M_1 + \cdots + M_\sigma = N)} \\ &= \frac{A_{3,m,\tau} A_{3,N-m,\sigma-\tau}}{A_{3,N,\sigma}} \\ &\approx \frac{\left(1+\frac{\tau-1}{2m}\right)^m \left(1+\frac{2m}{\tau-1}\right)^{(\tau-1)/2} \left(1+\frac{\sigma-\tau-2}{2(N-m)}\right)^{N-m} \left(1+\frac{2(N-m)}{\sigma-\tau-2}\right)^{(\sigma-\tau-2)/2} (N(2(N-m)+\sigma))^{1/2}}{\left(1+\frac{\sigma-1}{2N}\right)^N \left(1+\frac{2N}{\sigma-1}\right)^{(\sigma-1)/2} (2\pi m\sigma(N-m))^{1/2}} \\ &\le_b \frac{e^{(\tau-1)/2}\left(\left(\frac{2m}{\tau}\right)^{(\tau-1)/2} e^{o(\tau)}\right) e^{(\sigma-\tau)/2}\left(1+\frac{2(N-m)}{\sigma-\tau-2}\right)^{(\sigma-\tau-2)/2} (N(2(N-m)+\sigma))^{1/2}}{e^{\sigma/2-\sigma^2/8N}\left(\left(\frac{2N}{\sigma}\right)^{(\sigma-1)/2} e^{\sigma^2/(4+o(1))N}\right) (m\sigma(N-m))^{1/2}} \end{aligned}$$

$$\leq_b \frac{e^{o(\tau)}\left(\frac{2m}{\tau}\right)^{(\tau-1)/2}\left(1+\frac{2(N-m)}{\sigma-\tau-2}\right)^{(\sigma-\tau-2)/2}(N(2(N-m)+\sigma))^{1/2}}{\left(\frac{2N}{\sigma}\right)^{(\sigma-1)/2}(m\sigma(N-m))^{1/2}} \tag{4.1}$$

The above bound can be rewritten as

$$\leq_b \frac{e^{o(\tau)}\left(\frac{2}{\tau}\right)^{(\tau-1)/2}N^{1/2}\sigma^{(\sigma-1)/2}}{(2N)^{(\sigma-1)/2}\sigma^{1/2}} \times \frac{m^{(\tau-1)/2}\left(1+\frac{2(N-m)}{\sigma-\tau-2}\right)^{(\sigma-\tau-2)/2}(N-m+\sigma)^{1/2}}{(m(N-m))^{1/2}}.$$

Suppose first that $m \leq N-4\sigma$. Then, the bound becomes

$$\begin{aligned}
&\leq_b \frac{e^{o(\tau)}\left(\frac{2}{\tau}\right)^{(\tau-1)/2}N^{1/2}\sigma^{(\sigma-1)/2}}{(2N)^{(\sigma-1)/2}\sigma^{1/2}} \times m^{(\tau-2)/2}\left(1+\frac{2(N-m)}{\sigma-\tau-2}\right)^{(\sigma-\tau-2)/2} \qquad (4.2)\\
&\leq_b \frac{e^{o(\tau)}2^{(\tau-1)/2}N^{1/2}\sigma^{(\sigma-1)/2}}{(2N)^{(\sigma-1)/2}\tau^{\tau/2}} \times m^{(\tau-2)/2}\left(\frac{2(N-m)}{\sigma-\tau}\right)^{(\sigma-\tau)/2} e^{\sigma^2/(N-m)}\\
&\leq \frac{e^{o(\tau)}N^{1/2}}{m^{1/2}}\left(\frac{\sigma(N-m)}{N(\sigma-\tau)}\right)^{(\sigma-\tau)/2}\left(\frac{\sigma m}{\tau N}\right)^{(\tau-1)/2} e^{\sigma^2/(N-m)}\\
&\leq_b \frac{e^{o(\tau)}N^{1/2}}{m^{1/2}}\left(\frac{e^2 m\sigma}{\tau N}\cdot\exp\left\{-\frac{m(\sigma-\tau)}{(\tau-1)N}+\frac{2\sigma^2}{(\tau-1)(N-m)}\right\}\right)^{(\tau-1)/2}\\
&= \frac{e^{o(\tau)}N^{1/2}}{m^{1/2}}\left(\frac{e^2 m\sigma}{\tau N}\cdot\exp\left\{-\frac{m\sigma}{(\tau-1)N}\left(1-\frac{\tau}{\sigma}-\frac{2\sigma}{m}-\frac{2\sigma}{N-m}\right)\right\}\right)^{(\tau-1)/2}\\
&\leq \frac{e^{o(\tau)}N^{1/2}}{m^{1/2}}\left(\frac{e^2 m\sigma}{\tau N}\cdot\exp\left\{-\frac{m\sigma}{3\tau N}\right\}\right)^{(\tau-1)/2}.
\end{aligned}$$

We inflate this by $n^2\binom{2\sigma+1}{\tau}$ to account for our choices for $\sigma, \tau, T_1, \ldots, T_\tau$ to get

$$\leq_b n^2\frac{e^{o(\tau)}N^{1/2}}{m^{1/2}}\left(\frac{4e^4 m\sigma^3}{\tau^3 N}\cdot\exp\left\{-\frac{m\sigma}{3\tau N}\right\}\right)^{(\tau-1)/2}.$$

So, if $m_0 = \frac{100\tau N\log(\sigma/\tau)}{\sigma}$, then

$$\begin{aligned}
&\sum_{m=m_0}^{N-4\sigma} \mathbf{Pr}(\exists\sigma,\tau,T_1,\ldots,T_\tau : M_1+\cdots+M_\tau = m \mid M_1+\cdots+M_\sigma = N)\\
&\quad \leq_b n^2 e^{o(\tau)}N^{5/2}\sum_{m=m_0}^{N-4\sigma}\left(\frac{4e^4 m\sigma^3}{\tau^3 N}\cdot\exp\left\{-\frac{m\sigma}{3\tau N}\right\}\right)^{(\tau-1)/2}\\
&\quad \leq n^2 e^{o(\tau)}N^{7/2}\left(\frac{4e^4 m_0\sigma^3}{\tau^3 N}\cdot\exp\left\{-\frac{m_0\sigma}{3\tau N}\right\}\right)^{(\tau-1)/2},
\end{aligned}$$

since xe^{-Ax} is decreasing for $Ax \geq 1$,

$$= n^2 e^{o(\tau)} N^{7/2} \left(\frac{4e^4 m_0 \sigma}{\tau N} \exp\left\{-\frac{m_0\sigma}{6\tau N}\right\} \times \frac{\sigma^2}{\tau^2} \exp\left\{-\frac{m_0\sigma}{6\tau N}\right\}\right)^{(\tau-1)/2}$$

$$\leq n^2 N^{7/2} \left(400 e^{4+o(1)} \log\left(\frac{\sigma}{\tau}\right) \times e^{-50/3} \times \frac{\sigma^2}{\tau^2} \left(\frac{\tau}{\sigma}\right)^{50/3}\right)^{(\tau-1)/2}$$

$$= O(n^{-K}),$$

for any constant $K > 0$.

Suppose now that $N - 4\sigma \leq m \leq N - \sigma^{1/3}$. Then we can bound (4.2) by

$$\leq_b \frac{e^{o(\tau)} \left(\frac{2}{\tau}\right)^{(\tau-1)/2} \sigma^{(\sigma-1)/2}}{(2N)^{(\sigma-1)/2}} \times m^{(\tau-1)/2} e^{4\sigma}$$

$$\leq \left(\frac{e^8\sigma}{2N}\right)^{(\sigma-\tau)/2} \left(\frac{e^8\sigma}{\tau}\right)^{(\tau-1)/2}.$$

We inflate this by $n^2\binom{2\sigma+1}{\tau} < n^2 4^\sigma$ to get

$$\leq_b n^2 \left(\frac{8e^8\sigma}{N}\right)^{(\sigma-\tau)/2} \left(\frac{16e^8\sigma}{\tau}\right)^{(\tau-1)/2}.$$

So,

$$\sum_{m=N-4\sigma}^{N-\sigma^{1/3}} \mathbf{Pr}(\exists \sigma, \tau, T_1, \dots, T_\sigma : M_1 + \cdots + M_\tau = m \mid M_1 + \cdots + M_\sigma = N)$$

$$\leq_b n^2 N^2 \sigma \left(\frac{8e^8\sigma}{N}\right)^{(\sigma-\tau)/2} \left(\frac{16e^8\sigma}{\tau}\right)^{(\tau-1)/2} = O(n^{-K}),$$

for any constant $K > 0$, since $\sigma \log N \gg \tau \log \sigma$.

When $m \geq N - \sigma^{1/3}$ we replace (4.1) by

$$\leq_b \frac{\left(1+\frac{\tau-1}{2m}\right)^m \left(1+\frac{2m}{\tau-1}\right)^{(\tau-1)/2} \sigma^{N-m} N^{1/2}}{\left(1+\frac{\sigma-1}{2N}\right)^N \left(1+\frac{2N}{\sigma-1}\right)^{(\sigma-1)/2} (m\sigma)^{1/2}}$$

$$\leq_b \frac{e^{\tau/2+o(\tau)} \left(\frac{2m}{\tau}\right)^{(\tau-1)/2} \sigma^{N-m} N^{1/2}}{e^\sigma \left(\frac{2N}{\sigma}\right)^{(\sigma-1)/2} m^{1/2}}$$

$$\leq_b \left(\frac{e^{1+o(1)}\sigma}{\tau}\right)^{(\tau-1)/2} \left(\frac{\sigma}{2N}\right)^{(\sigma-\tau)/2} \sigma^{\sigma^{1/3}}.$$

Inflating this by $n^2 4^\sigma$ gives a bound of

$$\leq_b n^2 \left(\frac{16 e^{1+o(1)}\sigma}{\tau}\right)^{(\tau-1)/2} \left(\frac{8\sigma^{1+o(1)}}{N}\right)^{(\sigma-\tau)/2} = O(n^{-\text{any constant}}).$$

□

5. MODIFICATIONS OF THEOREM 1.1

Let $\lambda = \log^3 n$ and partition $[\lambda]$ into $q = \log n$ sets of size $\lambda_1 = \log^2 n$. Now add $n - \lambda$ vertices to $\mathcal{T}_\lambda$ and let M_i denote the number of vertices that land in the ith part Π_i of the partition. Lemma 4.1 implies that q.s.

$$M_i \leq M_{\max} = \frac{200n}{\log n} \log\log n, \quad 1 \leq i \leq \tau. \tag{5.1}$$

Let

$$\omega_1(x) = \log^{\alpha/2} x \tag{5.2}$$

for $x \in \mathbb{R}$.

Let L_i denote the length of the longest path in Π_i. Suppose that $\mathcal{T}_n$ contains a path of length at least $n/\omega_1,\ \omega_1 = \omega_1(n)$ and let k be the number of i such that

$$L_i \geq \frac{200n \log\log n}{\omega_1^2 \log n} \geq \frac{M_{\max}}{\log^\alpha(M_{\max})}.$$

Then, as $k \leq q = \log n$ we have

$$k\frac{200n \log\log n}{\log n} + (\log n - k)\frac{200n \log\log n}{\omega_1^2 \log n} \geq \frac{n}{\omega_1},$$

which implies that

$$k \geq \frac{\log n}{201\omega_1 \log\log n}.$$

Theorem 1.1 with the bound on M_i given in (5.1) implies that the probability of this is at most

$$\frac{1}{n} + \binom{\log n}{\frac{\log n}{201\omega_1 \log\log n}} \left(\frac{1}{\log^\alpha(n/\log n)}\right)^{\frac{\log n}{201\omega_1 \log\log n}} \leq \frac{1}{n} + \left(\frac{1}{\log^{\alpha/3} n}\right)^{\frac{\log n}{201\omega_1 \log\log n}} \leq \frac{1}{\phi(n,\omega_1)}, \tag{5.3}$$

where

$$\phi(x, y) = \exp\left\{\frac{\log x}{y \log\log x}\right\}.$$

The term $1/n$ accounts for the failure of the property in Lemma 4.1.

In summary, we have proved the following:

Lemma 5.1.

$$\mathbf{Pr}\left(L(n) \geq \frac{n}{\omega_1(n)}\right) \leq \frac{1}{\phi(n,\omega_1)}. \tag{5.4}$$

We are using $\phi(x, y)$ in place of $\phi(x)$ because we will need to use $\omega_1(x)$ for values of x other than n.

Next consider $\mathcal{A}_\sigma$ and $\lambda_1 \leq \tau \ll \sigma$ and let $T_1, T_2, \dots, T_\tau$ be a set of τ triangular faces of $\mathcal{A}_\sigma$. Suppose that we add $N \gg \sigma$ more vertices and let N_j be the number of vertices that are placed in T_j, $1 \leq j \leq \tau$.

Next let

$$\Lambda(x) = e^{x^2}, \tag{5.5}$$

where $x \in \mathbb{R}$.

Now let

$$J = \left\{ j : N_j \geq \Lambda_0 \right\}, \text{ where } \Lambda_0 = \Lambda(\omega_1(n)). \tag{5.6}$$

Let L_j denote the length of the longest path through the ApN defined by T_j and the N_j vertices it contains, $1 \leq j \leq \tau$. For the remainder of the section let

$$\omega_0 = \omega_1(\Lambda_0), \quad \phi_0 = \phi(\Lambda_0, \omega_0) = \exp\left\{ \frac{\omega_0}{2\log\omega_0} \right\}, \quad \omega_2 = \frac{\phi_0}{\omega_0}. \tag{5.7}$$

Then let

$$J_1 = \left\{ j \in J : L_j \geq \frac{N_j}{\omega_1(N_j)} \right\}. \tag{5.8}$$

We note that

$$\begin{aligned} \log\omega_2 = \log\phi_0 - \log\omega_0 &= \frac{\log\Lambda_0}{\omega_0 \log\log\Lambda} - \log\omega_0 \\ &= \frac{\omega_0^2}{(2 + o(1))\omega_0 \log\log\omega_0} - \log\omega_0. \end{aligned}$$

For $j \in J$, $N_j \geq \Lambda_0$ (see (5.6)). It follows from Lemma 5.1 that the size of J_1 is stochastically dominated by $Bin(\tau, 1/\phi_0)$. Using the bound

$$\mathbf{Pr}(Bin(n, p) \geq \alpha np) \leq \left(\frac{e}{\alpha}\right)^{\alpha np},$$

we find that

$$\mathbf{Pr}\left(|J_1| \geq \frac{\omega_2\tau}{\phi_0} \right) \leq \left(\frac{e}{\omega_2} \right)^{\omega_2\tau/\phi_0}. \tag{5.9}$$

Using this we prove

Lemma 5.2. *Suppose that*

$$\log\left(\frac{\sigma}{\tau}\right) \leq \frac{\omega_0}{\log\omega_0}.$$

Then q.s., for all $\lambda_1 \le \tau \ll \sigma \ll N$ *and all collections* $\mathcal{T}$ *of* τ *faces of* $\mathcal{A}_\sigma$ *we find that with* J_1 *as defined in* (5.8),

$$|J_1| \le \frac{\omega_2 \tau}{\phi_0}.$$

Proof. It follows from (5.9) that

$$\begin{aligned}
&\mathbf{Pr}\left(\exists \tau, \sigma, N, \mathcal{T} : |J_1| \ge \frac{\omega_2}{\tau \phi_0}\right) \\
&\le n^3 \binom{(2\sigma + 1)}{\tau} \left(\frac{e}{\omega_2}\right)^{\omega_2 \tau/\phi_0} \\
&\le n^3 \left(\frac{e(2\sigma + 1)}{\tau} \cdot \left(\frac{e}{\omega_2}\right)^{\omega_2/\phi_0}\right)^{\tau} \\
&\le \exp\left\{\tau \left(\frac{3 \log n}{\tau} + 2 + \log\left(\frac{\sigma}{\tau}\right) + \frac{\omega_2}{\phi_0} - \frac{\omega_2 \log \omega_2}{\phi_0}\right)\right\} \\
&\le \exp\left\{\tau \left(\frac{3 \log n}{\tau} + 2 + \frac{\omega_0}{\log \omega_0} + - \frac{\omega_0}{(2 + o(1)) \log \log \omega_0}\right)\right\} \\
&= O(n^{-\text{anyconstant}}).
\end{aligned}$$

□

6. PROOF OF THEOREM 1.2

Fix a path P of $\mathcal{A}_n$. Suppose that, after adding $\sigma \ge n^{1/2}$ vertices, we find that P visits

$$n^{1/2} \ge \tau \ge \lambda_1 \omega_0 \tag{6.1}$$

of the triangles $T_1, T_2, \ldots, T_\tau$ of $\mathcal{A}_\sigma$. Now consider adding N more vertices, where the value of N is given in (6.4) below. Let $\sigma' = \sigma + N$ and let τ' be the number of triangles of $\mathcal{A}_{\sigma'}$ that are visited by P.

We assume that

$$\frac{\alpha}{2} \log \log n \le \log\left(\frac{\sigma}{\tau}\right) \le \frac{\omega_0}{\log \omega_0}. \tag{6.2}$$

Let M_i be the number of vertices placed in T_i and let N_i be the number of these that are visited by P. It follows from Lemma 4.1 that w.h.p.

$$\sum_{i=1}^{\tau} M_i \le \frac{100 \tau N}{\sigma} \log\left(\frac{\sigma}{\tau}\right).$$

Now w.h.p.,

$$\sum_{i=1}^{\tau} N_i \le \tau\Lambda_0 + \frac{100\omega_2\tau N}{\phi_0\sigma}\log\left(\frac{\sigma\phi_0}{\omega_2\tau}\right) + \frac{100\tau N}{\sigma\omega_0}\log\left(\frac{\sigma}{\tau}\right). \tag{6.3}$$

To see (6.3), observe that $\tau\Lambda_0$ bounds the contribution from $[\tau]\setminus J$ (see (5.6)). The second term bounds the contribution from J_1. Now $|J_1| < \omega_2\tau/\phi_0 \ll \tau$ as shown in Lemma 5.2. We cannot apply Lemma 4.1 to bound the contribution of J_1 unless we know that $|J_1| \ge \lambda_1$. We choose an arbitrary set of indices $J_2 \subseteq [\tau]\setminus J_1$ of size $\omega_2\tau/\phi_0 - |J_1|$ and then the middle term bounds the contribution of $J_1 \cup J_2$. Note that $\omega_2\tau/\phi_0 = \tau/\omega_0 \ge \lambda_1$ from (6.1). The third term bounds the contribution from $J \setminus J_1$. Here, we use $\omega_1(N_j) \ge \omega_1(\Lambda_0) = \omega_0$; see (5.8).

We now choose

$$N = 3\sigma\Lambda_0. \tag{6.4}$$

We observe that

$$\frac{\omega_2}{\phi_0}\log\left(\frac{\sigma\phi_0}{\omega_2\tau}\right) \le \frac{1}{\omega_0}\left(\frac{\omega_0}{\log\omega_0} + 2\log\omega_0\right) = o(1).$$

$$\frac{1}{\omega_0}\log\left(\frac{\sigma}{\tau}\right) \le \frac{1}{\log\omega_0} = o(1).$$

Now, along with Lemma 3.1, this implies that

$$\tau' \le \sum_{i=1}^{\tau}(N_i+1) \le \tau + \tau\Lambda_0 + o\left(\frac{\tau N}{\sigma}\right).$$

Since $\sigma' = \sigma + N$, this implies that

$$\frac{\tau'}{\sigma'} \le \left(\frac{1}{3} + o(1)\right)\frac{\tau}{\sigma} < \frac{\tau}{2\sigma}.$$

It follows by repeated application of this argument that we can replace Theorem 1.1 by the following lemma:

Lemma 6.1.

$$\mathbf{Pr}\left(L(n) \ge \log n + \frac{100\log n}{e^{\omega_0/\log\omega_0}}n\right) = O\left(\frac{1}{\phi(n,\omega_1(n))}\right).$$

Proof. We add the vertices in rounds of size $\sigma_0 = n^{1/2}, \sigma_1, \ldots, \sigma_m$. Here, $\sigma_i = 3\sigma_{i-1}\Lambda_0$ and $m-1 \ge (1-o(1))\frac{\log n}{\log\Lambda_0} = (1-o(1))\frac{\log n}{\omega_1(n)^2} = \log^{1-2\alpha} n$. We let $P_0, P_1, P_2, \ldots, P_m = P$ be a sequence of paths, where P_i is a path in $\mathcal{A}_i = \mathcal{A}_{\sigma_0+\cdots+\sigma_i}$. Furthermore, P_i is obtained from P_{i+1} in the same way that Q is obtained from P in Lemma 3.1. We let τ_i denote the number of faces of $\mathcal{A}_i$ whose interior is visited by P_i. It follows from Lemma 3.1 and Lemma 4.1 that the length of P is bounded by

$$m + \frac{\tau_{m-1}}{\sigma_{m-1}} \sigma_m \log\left(\frac{\sigma_{m-1}}{\tau_{m-1}}\right),$$

since the second term is a bound on the number of points in the interior of triangles of $\mathcal{A}_{m-1}$ visited by P.

We have w.h.p. that

$$\frac{\sigma_i}{\tau_i} \geq \begin{cases} \frac{2\sigma_{i-1}}{\tau_{i-1}} & \frac{\sigma_{i-1}}{\tau_{i-1}} \leq e^{\omega_0/\log\omega_0} \\ \frac{\sigma_{i-1}}{100\tau_{i-1}\log(\sigma_{i-1}/\tau_{i-1})} & \frac{\sigma_{i-1}}{\tau_{i-1}} > e^{\omega_0/\log\omega_0} \end{cases}.$$

The second inequality here is from Lemma 4.1.

The result follows from $2^{\log^{1-2\alpha} n} \geq e^{\omega_0/\log\omega_0}$. □

To get Theorem 1.2, we repeat the argument in Sections 5 and 6, but we start with $\omega_1(x) = \log^{1/3} x$. The claim in Theorem 1.2 is then slightly weaker than the claim in Lemma 6.1.

We note that subsequent to completion of this paper, Theorem 1.2 has been improved [2] with a high probability upper bound of $O(n^{1-\varepsilon})$ on $L(n)$.

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